2004 AMC 10B Problems/Problem 1

Problem

Each row of the Misty Moon Amphitheater has 33 seats. Rows 12 through 22 are reserved for a youth club. How many seats are reserved for this club?

$\mathrm{(A)}\ 297\qquad\mathrm{(B)}\ 330\qquad\mathrm{(C)}\ 363\qquad\mathrm{(D)}\ 396\qquad\mathrm{(E)}\ 726$

Solution

There are $22-12+1=11$ rows of 33 seats, giving $11\times 33=\boxed{\mathrm{(C)}\ 363}$ seats.

See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 10B Problems/Problem 2

Contents

Problem

How many two-digit positive integers have at least one 7 as a digit?

$\mathrm{(A)}\ 10 \qquad \mathrm{(B)}\ 18 \qquad \mathrm{(C)}\ 19 \qquad \mathrm{(D)}\ 20 \qquad \mathrm{(E)}\ 30$

Solution 1

Ten numbers $(70, 71, \dots, 79)$ have 7 as the tens digit. Nine numbers $(17, 27, \dots, 97)$ have it as the ones digit. Number 77 is in both sets.

Thus the result is $10 + 9 - 1 = 18 \Rightarrow \boxed{\mathrm{(B)}\ 18}$.

Solution 2

We use complementary counting. The complement of having at least one 7 as a digit is having no 7s as a digit.

We have 9 digits to choose from for the first digit and 10 digits for the second. This gives a total of $9 \times 10 = 90$ two-digit numbers.

But since we cannot have 7 as a digit, we have 8 first digits and 9 second digits to choose from.

Thus there are $8 \times 9 = 72$ two-digit numbers without a 7 as a digit.

90 (The total number of two-digit numbers) -72 (The number of two-digit numbers without a 7) $= 18 \Rightarrow \boxed{\mathrm{(B)}\ 18}$.

See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 10B Problems/Problem 3

Problem

At each basketball practice last week, Jenny made twice as many free throws as she made at the previous practice. At her fifth practice she made 48 free throws. How many free throws did she make at the first practice?

$\mathrm{(A) \ } 3 \qquad \mathrm{(B) \ } 6 \qquad \mathrm{(C) \ } 9 \qquad \mathrm{(D) \ } 12 \qquad \mathrm{(E) \ } 15$

Solution

At the fourth practice she made $48/2=24$ throws, at the third one it was $24/2=12$, then we get $12/2=6$ throws for the second practice, and finally $6/2=3 \Rightarrow \boxed{\mathrm{(A)}\ 3}$ throws at the first one.

See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 10B Problems/Problem 4

Contents

Problem

A standard six-sided die is rolled, and P is the product of the five numbers that are visible. What is the largest number that is certain to divide P?

$\mathrm{(A) \ } 6 \qquad \mathrm{(B) \ } 12 \qquad \mathrm{(C) \ } 24 \qquad \mathrm{(D) \ } 144 \qquad \mathrm{(E) \ } 720$

Solution 1

The product of all six numbers is $6! = 720$. The products of numbers that can be visible are $720/1$, $720/2$, ..., $720/6$. The answer to this problem is their greatest common divisor -- which is $720/L$, where L is the least common multiple of $\{1,2,3,4,5,6\}$. Clearly $L=60$ and the answer is $720/60 = \boxed{\mathrm{(B)}\ 12}$.

Solution 2

Clearly, P cannot have a prime factor other than 2, 3 and 5.

We can not guarantee that the product will be divisible by 5, as the number 5 can end on the bottom.

We can guarantee that the product will be divisible by 3 (one of 3 and 6 will always be visible), but not by 3^2.

Finally, there are three even numbers, hence two of them are always visible and thus the product is divisible by 2^2. This is the most we can guarantee, as when the 4 is on the bottom side, the two visible even numbers are 2 and 6, and their product is not divisible by 2^3.

Hence $P = 3\cdot 2^2 = \boxed{\mathrm{(B)}\ 12}$.

See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 12B Problems/Problem 2

The following problem is from both the 2004 AMC 12B #2 and 2004 AMC 10B #5, so both problems redirect to this page.

Problem 2

In the expression $c\cdot a^b-d$, the values of a, b, c, and d are 0, 1, 2, and 3, although not necessarily in that order. What is the maximum possible value of the result?

$\mathrm{(A)}\ 5\qquad\mathrm{(B)}\ 6\qquad\mathrm{(C)}\ 8\qquad\mathrm{(D)}\ 9\qquad\mathrm{(E)}\ 10$

Solution

If $a=0$ or $c=0$, the expression evaluates to $-d<0$.
If $b=0$, the expression evaluates to $c-d\le 2$.
Case $d=0$ remains. In that case, we want to maximize $c\cdot a^b$ where $\{a,b,c\}=\{1,2,3\}$. Trying out the six possibilities we get that the best one is $(a,b,c)=(3,2,1)$, where $c\cdot a^b=1\cdot 3^2=\boxed{\mathrm{(D)}\ 9}$.

See Also

2004 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2004))	
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2004 AMC 10B Problems/Problem 6

Problem

Which of the following numbers is a perfect square?

$\mathrm{(A)}\ 98!\cdot 99! \qquad \mathrm{(B)}\ 98!\cdot 100! \qquad \mathrm{(C)}\ 99!\cdot 100! \qquad \mathrm{(D)}\ 99!\cdot 101! \qquad \mathrm{(E)}\ 100!\cdot 101!$

Solution

Using the fact that $n! = n\cdot(n-1)!$, we can write:

$$A = 98!\cdot(99\cdot 98!) = 99\cdot(98!)^2 = 11\cdot 3^2\cdot(98!)^2 \qquad (1)$$
$$B = 100\cdot 99\cdot(98!)^2 = 11\cdot 10^2\cdot 3^2\cdot(98!)^2 \qquad (2)$$
$$C = 100\cdot(99!)^2 = 10^2\cdot(99!)^2 \qquad (3)$$
$$D = 101\cdot 100\cdot(99!)^2 = 101\cdot 10^2\cdot(99!)^2 \qquad (4)$$
$$E = 101\cdot(100!)^2 \qquad (5)$$

We see that $\boxed{\mathrm{(C)}\ 99!\cdot 100!}$ is a square, and because 11, and 101 are primes, none of the other four choices are squares.

See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 12B Problems/Problem 5

The following problem is from both the 2004 AMC 12B #5 and 2004 AMC 10B #7, so both problems redirect to this page.

Contents

Problem

On a trip from the United States to Canada, Isabella took d U.S. dollars. At the border she exchanged them all, receiving 10 Canadian dollars for every 7 U.S. dollars. After spending 60 Canadian dollars, she had d Canadian dollars left. What is the sum of the digits of d?

$\mathrm{(A)\ }5\qquad\mathrm{(B)\ }6\qquad\mathrm{(C)\ }7\qquad\mathrm{(D)\ }8\qquad\mathrm{(E)\ }9$

Solution 1

Isabella had $60+d$ Canadian dollars. Setting up an equation we get $d=\frac{7}{10}\cdot(60+d)$, which solves to $d=140$, and the sum of digits of d is $\boxed{\mathrm{(A)\ }5}$.

Solution 2

Each time Isabella exchanges 7 U.S. dollars, she gets 7 Canadian dollars and 3 Canadian dollars extra. Isabella received a total of 60 Canadian dollars extra, therefore she exchanged 7 U.S. dollars $\frac{60}{3}=20$ times. Thus $d=7\cdot 20=\boxed{\mathrm{(A)\ }5}$.

See Also

2004 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2004))	
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2004 AMC 12B Problems/Problem 6

The following problem is from both the 2004 AMC 12B #6 and 2004 AMC 10B #8, so both problems redirect to this page.

Problem

Minneapolis-St. Paul International Airport is 8 miles southwest of downtown St. Paul and 10 miles southeast of downtown Minneapolis. Which of the following is closest to the number of miles between downtown St. Paul and downtown Minneapolis?

$\mathrm{(A)}\ 13\qquad\mathrm{(B)}\ 14\qquad\mathrm{(C)}\ 15\qquad\mathrm{(D)}\ 16\qquad\mathrm{(E)}\ 17$

Solution

The directions "southwest" and "southeast" are orthogonal. Thus the described situation is a right triangle with legs 8 miles and 10 miles long. The hypotenuse length is $\sqrt{8^2+10^2}\approx 12.8$, and thus the answer is $\boxed{\mathrm{(A)}\ 13}$.

Without a calculator one can note that $8^2+10^2=164<169=13^2\Rightarrow\boxed{\mathrm{(A)}\ 13}$.

See Also

2004 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2004))	
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2004 AMC 12B Problems/Problem 7

The following problem is from both the 2004 AMC 12B #7 and 2004 AMC 10B #9, so both problems redirect to this page.

Problem

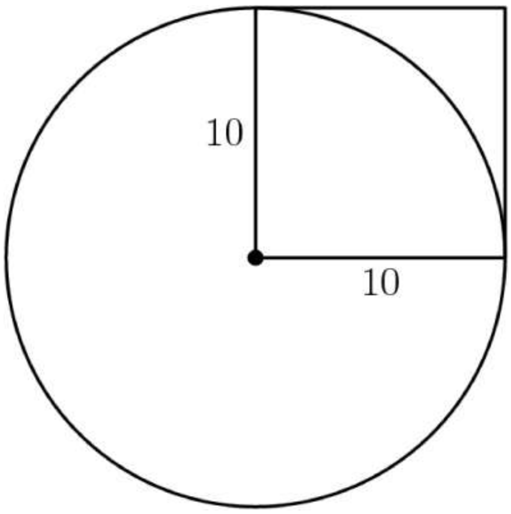
A square has sides of length 10, and a circle centered at one of its vertices has radius 10. What is the area of the union of the regions enclosed by the square and the circle?

$\mathrm{(A)\ } 200+25\pi \qquad \mathrm{(B) \ } 100+75\pi \qquad \mathrm{(C) \ } 75+100\pi \qquad \mathrm{(D) \ } 100+100\pi \qquad \mathrm{(E) \ } 100+125\pi$

Solution

The area of the circle is $S_{\bigcirc}=100\pi$; the area of the square is $S_{\square}=100$.

Exactly $\frac{1}{4}$ of the circle lies inside the square. Thus the total area is $\frac{3}{4}S_{\bigcirc}+S_{\square}=\boxed{\mathrm{(B)\ }100+75\pi}$.



See Also

2004 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2004))	
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2004 AMC 12B Problems/Problem 8

The following problem is from both the 2004 AMC 12B #8 and 2004 AMC 10B #10, so both problems redirect to this page.

Problem

A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

$\mathrm{(A)}\ 5\qquad\mathrm{(B)}\ 8\qquad\mathrm{(C)}\ 9\qquad\mathrm{(D)}\ 10\qquad\mathrm{(E)}\ 11$

Solution

The sum of the first n odd numbers is n^2. As in our case $n^2=100$, we have $n=\boxed{\mathrm{(D)}\ 10}$.

See Also

2004 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2004))	
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2004 AMC 10B Problems/Problem 11

Contents

Problem

Two eight-sided dice each have faces numbered 1 through 8. When the dice are rolled, each face has an equal probability of appearing on the top. What is the probability that the product of the two top numbers is greater than their sum?

$\mathrm{(A)}\ \frac{1}{2}\qquad\mathrm{(B)}\ \frac{47}{64}\qquad\mathrm{(C)}\ \frac{3}{4}\qquad\mathrm{(D)}\ \frac{55}{64}\qquad\mathrm{(E)}\ \frac{7}{8}$

Solution 1

We have $1\times n = n < 1 + n$, hence if at least one of the numbers is 1, the sum is larger. There 15 such possibilities.

We have $2\times 2 = 2 + 2$.

For $n>2$ we already have $2\times n = n + n > 2 + n$, hence all other cases are good.

Out of the 8×8 possible cases, we find that in $15+1=16$ the sum is greater than or equal to the product, hence in $64-16=48$ cases the sum is smaller, satisfying the condition. Therefore the answer is $\frac{48}{64} = \boxed{\mathrm{(C)}\ \frac{3}{4}}$.

Solution 2

Let the two rolls be m, and n.

From the restriction: $mn > m + n$

$mn - m - n > 0$

$mn - m - n + 1 > 1$

$(m-1)(n-1) > 1$

Since $m-1$ and $n-1$ are non-negative integers between 1 and 8, either $(m-1)(n-1) = 0$, $(m-1)(n-1) = 1$, or $(m-1)(n-1) > 1$

$(m-1)(n-1) = 0$ if and only if $m=1$ or $n=1$.

There are 8 ordered pairs (m,n) with $m=1$, 8 ordered pairs with $n=1$, and 1 ordered pair with $m=1$ and $n=1$. So, there are $8+8-1=15$ ordered pairs (m,n) such that $(m-1)(n-1) = 0$.

$(m-1)(n-1) = 1$ if and only if $m-1=1$ and $n-1=1$ or equivalently $m=2$ and $n=2$. This gives 1 ordered pair $(m,n) = (2,2)$.

So, there are a total of $15+1=16$ ordered pairs (m,n) with $(m-1)(n-1) < 1$.

Since there are a total of $8\cdot8=64$ ordered pairs (m,n), there are $64-16=48$ ordered pairs (m,n) with $(m-1)(n-1) > 1$.

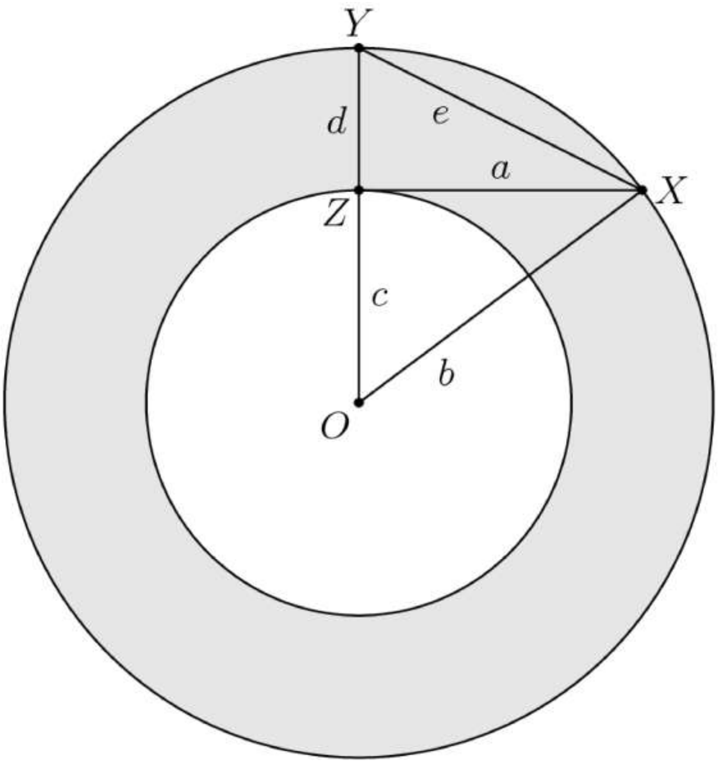
Thus, the desired probability is $\frac{48}{64} = \boxed{\mathrm{(C)}\ \frac{3}{4}}$.

2004 AMC 12B Problems/Problem 10

The following problem is from both the 2004 AMC 12B #10 and 2004 AMC 10B #12, so both problems redirect to this page.

Problem

An annulus is the region between two concentric circles. The concentric circles in the figure have radii b and c, with $b > c$. Let OX be a radius of the larger circle, let XZ be tangent to the smaller circle at Z, and let OY be the radius of the larger circle that contains Z. Let $a = XZ$, $d = YZ$, and $e = XY$. What is the area of the annulus?



$\textbf{(A)}\ \pi a^2 \qquad \textbf{(B)}\ \pi b^2 \qquad \textbf{(C)}\ \pi c^2 \qquad \textbf{(D)}\ \pi d^2 \qquad \textbf{(E)}\ \pi e^2$

Solution

The area of the large circle is πb^2, the area of the small one is πc^2, hence the shaded area is $\pi(b^2 - c^2)$.

From the Pythagorean Theorem for the right triangle OXZ we have $a^2 + c^2 = b^2$, hence $b^2 - c^2 = a^2$ and thus the shaded area is $\boxed{\textbf{(A)}\ \pi a^2}$.

See also

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2004 AMC 10B Problems/Problem 13

Contents

Problem

In the United States, coins have the following thicknesses: penny, 1.55 mm; nickel, 1.95 mm; dime, 1.35 mm; quarter, 1.75 mm. If a stack of these coins is exactly 14 mm high, how many coins are in the stack?

$\mathrm{(A) \ } 7 \qquad \mathrm{(B) \ } 8 \qquad \mathrm{(C) \ } 9 \qquad \mathrm{(D) \ } 10 \qquad \mathrm{(E) \ } 11$

Solution

All numbers in this solution will be in hundredths of a millimeter.

The thinnest coin is the dime, with thickness 135. A stack of n dimes has height $135n$.

The other three coin types have thicknesses $135+20$, $135+40$, and $135+60$. By replacing some of the dimes in our stack by other, thicker coins, we can clearly create exactly all heights in the set $\{135n,135n+20,135n+40,\dots,195n\}$.

If we take an odd n, then all the possible heights will be odd, and thus none of them will be 1400. Hence n is even.

If $n<8$ the stack will be too low and if $n>10$ it will be too high. Thus we are left with cases $n=8$ and $n=10$.

If $n=10$ the possible stack heights are $1350,1370,1390,\dots$, with the remaining ones exceeding 1400.

Therefore there are $\boxed{\mathrm{(B) \ } 8}$ coins in the stack.

Using the above observation we can easily construct such a stack. A stack of 8 dimes would have height $8\cdot 135 = 1080$, thus we need to add 320. This can be done for example by replacing five dimes by nickels (for $+60\cdot 5 = +300$), and one dime by a penny (for $+20$).

Note

We can easily add up 1.55 mm and 1.95 mm to get 3.50 mm. We multiply that by 4 to get 14 mm. Since this works and it requires 8 coins, the answer is clearly $\boxed{\mathrm{(B) \ } 8}$.

See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 10B Problems/Problem 14

Problem

A bag initially contains red marbles and blue marbles only, with more blue than red. Red marbles are added to the bag until only $\frac{1}{3}$ of the marbles in the bag are blue. Then yellow marbles are added to the bag until only $\frac{1}{5}$ of the marbles in the bag are blue. Finally, the number of blue marbles in the bag is doubled. What fraction of the marbles now in the bag are blue?

$\mathrm{(A)}\ \frac{1}{5} \qquad \mathrm{(B)}\ \frac{1}{4} \qquad \mathrm{(C)}\ \frac{1}{3} \qquad \mathrm{(D)}\ \frac{2}{5} \qquad \mathrm{(E)}\ \frac{1}{2}$

Solution

We can ignore most of the problem statement. The only important information is that immediately before the last step blue marbles formed $\frac{1}{5}$ of the marbles in the bag. This means that there were x blue and $4x$ other marbles, for some x. When we double the number of blue marbles, there will be $2x$ blue and $4x$ other marbles, hence blue marbles now form $\boxed{\mathrm{(C)}\ \frac{1}{3}}$ of all marbles in the bag.

See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 10B Problems/Problem 15

Contents

Problem

Patty has 20 coins consisting of nickels and dimes. If her nickels were dimes and her dimes were nickels, she would have 70 cents more. How much are her coins worth?

$\textbf{(A)}\ \$1.15\qquad\textbf{(B)}\ \$1.20\qquad\textbf{(C)}\ \$1.25\qquad\textbf{(D)}\ \$1.30\qquad\textbf{(E)}\ \$1.35$

Solution 1

She has n nickels and $d=20-n$ dimes. Their total cost is $5n+10d=5n+10(20-n)=200-5n$ cents. If the dimes were nickels and vice versa, she would have $10n+5d=10n+5(20-n)=100+5n$ cents. This value should be 70 cents more than the previous one. We get $200-5n+70=100+5n$, which solves to $n=17$. Her coins are worth $200-5n=\boxed{\mathrm{(A)}\ \$1.15}$.

Solution 2

Changing a nickel into a dime increases the sum by 5 cents, and changing a dime into a nickel decreases it by the same amount. As the sum increased by 70 cents, there are $70/5=14$ more nickels than dimes. As the total count is 20, this means that there are 17 nickels and 3 dimes, which is equal to $\boxed{\mathrm{(A)}\ \$1.15}$.

See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 10B Problems/Problem 16

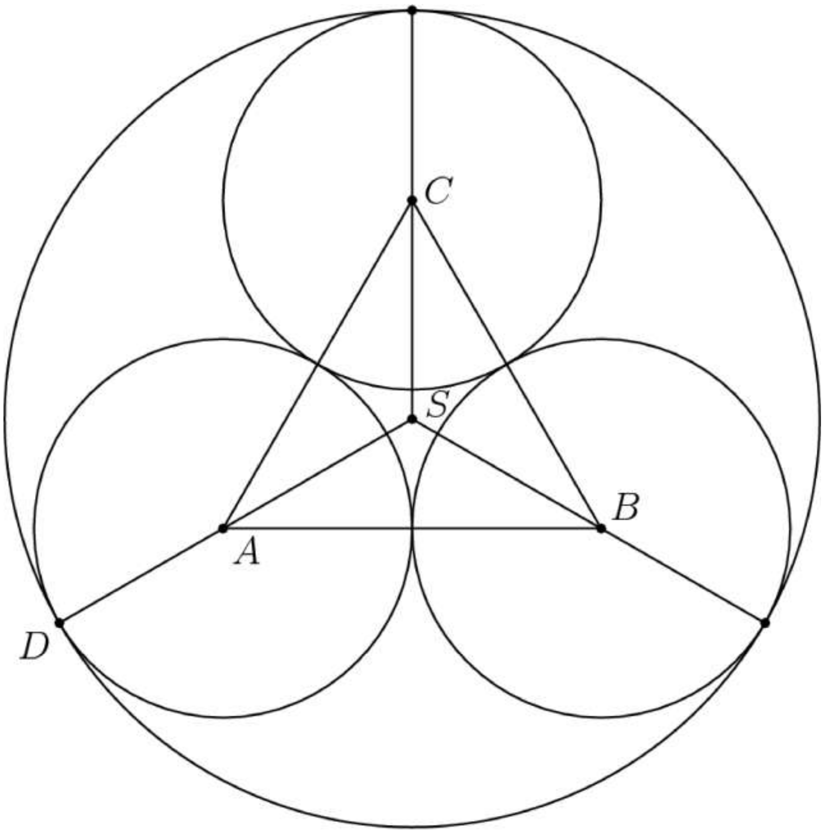
Problem

Three circles of radius 1 are externally tangent to each other and internally tangent to a larger circle. What is the radius of the large circle?

$\mathrm{(A)}\ \frac{2+\sqrt{6}}{3} \qquad \mathrm{(B)}\ 2 \qquad \mathrm{(C)}\ \frac{2+3\sqrt{2}}{2} \qquad \mathrm{(D)}\ \frac{3+2\sqrt{3}}{3} \qquad \mathrm{(E)}\ \frac{3+\sqrt{3}}{2}$

Solution

The situation is shown in the picture below. The radius we seek is $SD=AD+AS$. Clearly $AD=1$. The point S is clearly the center of the equilateral triangle ABC, thus AS is $2/3$ of the altitude of this triangle. We get that $AS=\frac{2}{3}\cdot\sqrt{3}$. Therefore the radius we seek is $1+\frac{2}{3}\cdot\sqrt{3}=\boxed{\mathrm{(D)}\ \frac{3+2\sqrt{3}}{3}}$.



See also

2004 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2004))	
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2004 AMC 12B Problems/Problem 15

The following problem is from both the 2004 AMC 12B #15 and 2004 AMC 10B #17, so both problems redirect to this page.

Contents

Problem

The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years Jack will be twice as old as Bill will be then. What is the difference in their current ages?

$\mathrm{(A) \ } 9 \qquad \mathrm{(B) \ } 18 \qquad \mathrm{(C) \ } 27 \qquad \mathrm{(D) \ } 36 \qquad \mathrm{(E) \ } 45$

Solution 1

If Jack's current age is $\overline{ab}=10a+b$, then Bill's current age is $\overline{ba}=10b+a$.

In five years, Jack's age will be $10a+b+5$ and Bill's age will be $10b+a+5$.

We are given that $10a+b+5=2(10b+a+5)$. Thus $8a=19b+5$.

For $b=1$ we get $a=3$. For $b=2$ and $b=3$ the value $\frac{19b+5}{8}$ is not an integer, and for $b\geq 4$ it is more than 9. Thus the only solution is $(a,b)=(3,1)$, and the difference in ages is $31-13=\boxed{\mathrm{(B)}\ 18}$.

Solution 2

Age difference does not change in time. Thus in five years Bill's age will be equal to their age difference.

The age difference is $(10a+b)-(10b+a)=9(a-b)$, hence it is a multiple of 9. Thus Bill's current age modulo 9 must be 4.

Thus Bill's age is in the set $\{13,22,31,40,49,58,67,76,85,94\}$.

As Jack is older, we only need to consider the cases where the tens digit of Bill's age is smaller than the ones digit. This leaves us with the options $\{13,49,58,67\}$.

Checking each of them, we see that only 13 works, and gives the solution $31-13=\boxed{\mathrm{(B)}\ 18}$.

See also

2004 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2004))	
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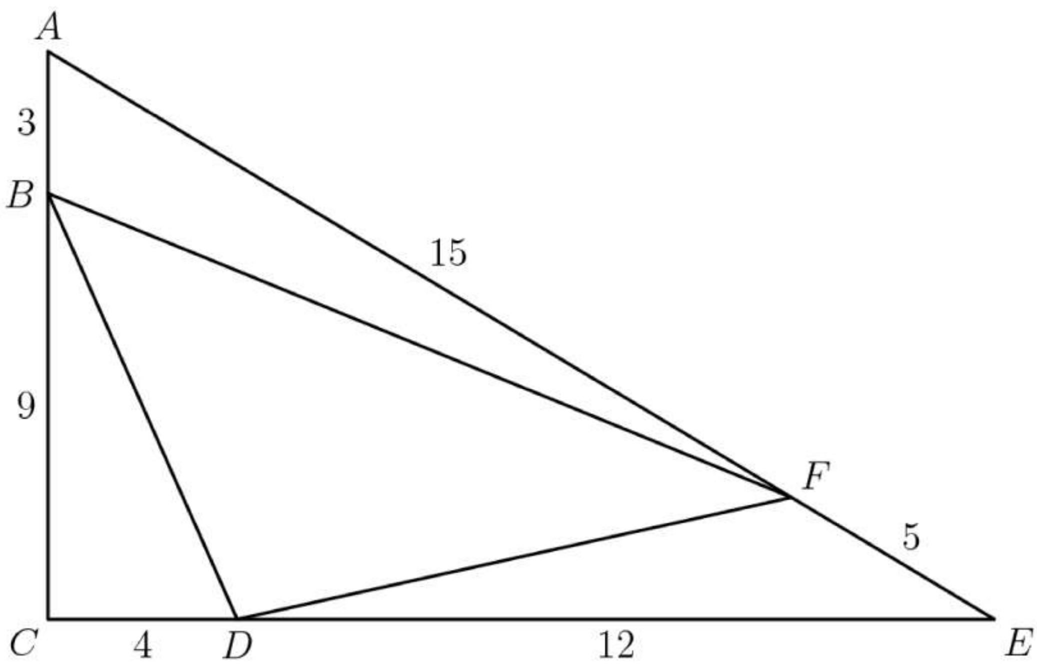
2004 AMC 10B Problems/Problem 18

Contents

Problem

In the right triangle $\triangle ACE$, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of $\triangle DBF$ to that of $\triangle ACE$?

$\text{(A)}\ \frac{1}{4} \qquad \text{(B)}\ \frac{9}{25} \qquad \text{(C)}\ \frac{3}{8} \qquad \text{(D)}\ \frac{11}{25} \qquad \text{(E)}\ \frac{7}{16}$



Solution 1

Let $x = [DBF]$. Because $\triangle ACE$ is divided into four triangles, $[ACE] = [BCD] + [ABF] + [DEF] + x$.

Because of SAS triangle area,
$\frac{1}{2} \cdot 12 \cdot 16 = \frac{1}{2} \cdot 9 \cdot 4 + \frac{1}{2} \cdot 3 \cdot 15 \cdot \sin(\angle A) + \frac{1}{2} \cdot 5 \cdot 12 \cdot \sin(\angle E) + x.$

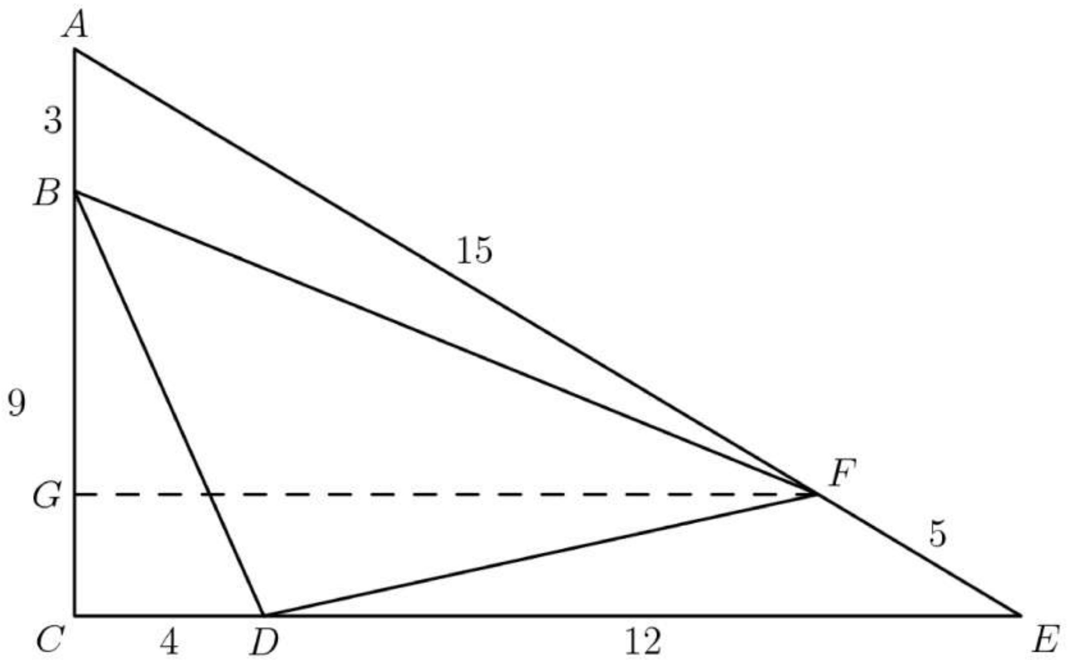
$\sin(\angle A) = \frac{16}{20}$ and $\sin(\angle E) = \frac{12}{20}$, so $96 = 18 + 18 + 18 + x.$

$x = 42$, so $\frac{[DBF]}{[ACE]} = \frac{42}{96} = \boxed{\textbf{(E)} \frac{7}{16}}$.

Solution 2

First of all, note that $\frac{AB}{AC} = \frac{CD}{CE} = \frac{EF}{EA} = \frac{1}{4}$, and therefore $\frac{BC}{AC} = \frac{DE}{CE} = \frac{FA}{EA} = \frac{3}{4}$.

Draw the height from F onto AB as in the picture below:



Now consider the area of $\triangle ABF$. Clearly the triangles $\triangle AFG$ and $\triangle AEC$ are similar, as they have all angles equal. Their ratio is $\frac{AF}{AE} = \frac{3}{4}$, hence $FG = \frac{3}{4} \cdot CE$. Now the area S_{ABF} of $\triangle ABF$ can be computed as $S_{ABF} = \frac{1}{2} \cdot AB \cdot FG = \frac{1}{2} \cdot \left(\frac{1}{4} \cdot AC\right) \cdot \left(\frac{3}{4} \cdot EC\right) = \frac{1}{4} \cdot \frac{3}{4} \cdot S_{ACE}$.

Similarly we can find that $S_{BCD} = S_{DEF} = \frac{3}{16} \cdot S_{ACE}$ as well.

Hence $S_{BDF} = S_{ACE} - 3 \cdot \left(\frac{3}{16} \cdot S_{ACE}\right) = \frac{7}{16} \cdot S_{ACE}$, and the answer is $\frac{S_{BDF}}{S_{ACE}} = \boxed{\frac{7}{16}}$.

See also

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2004 AMC 12B Problems/Problem 12

The following problem is from both the 2004 AMC 12B #12 and 2004 AMC 10B #19, so both problems redirect to this page.

Contents

Problem

In the sequence $2001, 2002, 2003, \ldots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the $2004^\textrm{th}$ term in this sequence?

$\mathrm{(A)}\ -2004 \qquad \mathrm{(B)}\ -2 \qquad \mathrm{(C)}\ 0 \qquad \mathrm{(D)}\ 4003 \qquad \mathrm{(E)}\ 6007$

Solution

Solution 1

We already know that $a_1 = 2001$, $a_2 = 2002$, $a_3 = 2003$, and $a_4 = 2000$. Let's compute the next few terms to get the idea how the sequence behaves. We get $a_5 = 2002 + 2003 - 2000 = 2005$, $a_6 = 2003 + 2000 - 2005 = 1998$, $a_7 = 2000 + 2005 - 1998 = 2007$, and so on.

We can now discover the following pattern: $a_{2k+1} = 1999 + 2k$ and $a_{2k} = 2004 - 2k$. This is easily proved by induction. It follows that $a_{2004} = a_{2\cdot 1002} = 2004 - 2\cdot 1002 = \boxed{0}$.

Solution 2

Note that the recurrence $a_n + a_{n+1} - a_{n+2} \quad = \quad a_{n+3}$ can be rewritten as $a_n + a_{n+1} \quad = \quad a_{n+2} + a_{n+3}$.

Hence we get that $a_1 + a_2 \quad = \quad a_3 + a_4 \quad = \quad a_5 + a_6 \quad = \cdots$ and also $a_2 + a_3 \quad = \quad a_4 + a_5 \quad = \quad a_6 + a_7 \quad = \cdots$

From the values given in the problem statement we see that $a_3 = a_1 + 2$.

From $a_1 + a_2 = a_3 + a_4$ we get that $a_4 = a_2 - 2$.

From $a_2 + a_3 = a_4 + a_5$ we get that $a_5 = a_3 + 2$.

Following this pattern, we get $a_{2004} = a_{2002} - 2 = a_{2000} - 4 = \cdots = a_2 - 2002 = \boxed{0}$.

See also

2004 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2004))	
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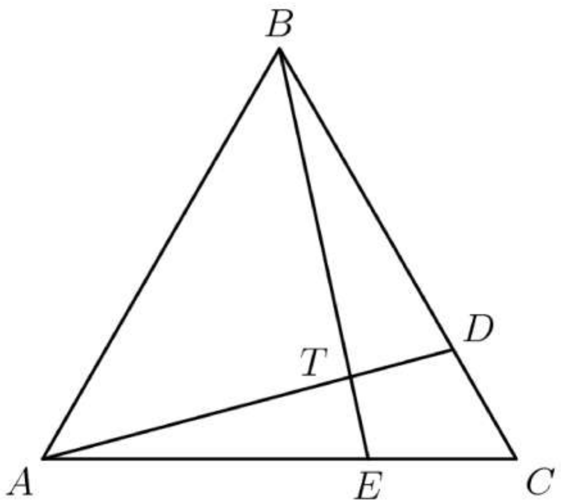
2004 AMC 10B Problems/Problem 20

Contents

Problem

In $\triangle ABC$ points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $AT/DT = 3$ and $BT/ET = 4$, what is CD/BD?

$$\text{(A) } \frac{1}{8} \qquad \text{(B) } \frac{2}{9} \qquad \text{(C) } \frac{3}{10} \qquad \text{(D) } \frac{4}{11} \qquad \text{(E) } \frac{5}{12}$$



Solution (Triangle Areas)

We use the square bracket notation $[\cdot]$ to denote area.

Without loss of generality, we can assume $[\triangle BTD] = 1$. Then $[\triangle BTA] = 3$, and $[\triangle ATE] = 3/4$. We have $CD/BD = [\triangle ACD]/[\triangle ABD]$, so we need to find the area of quadrilateral $TDCE$.

Draw the line segment TC to form the two triangles $\triangle TDC$ and $\triangle TEC$. Let $x = [\triangle TDC]$, and $y = [\triangle TEC]$. By considering triangles $\triangle BTC$ and $\triangle ETC$, we obtain $(1+x)/y = 4$, and by considering triangles $\triangle ATC$ and $\triangle DTC$, we obtain $(3/4+y)/x = 3$. Solving, we get $x = 4/11$, $y = 15/44$, so the area of quadrilateral $TDEC$ is $x + y = 31/44$.

Therefore $\dfrac{CD}{BD} = \dfrac{\frac{3}{4}+\frac{31}{44}}{3+1} = \boxed{\textbf{(D)} \frac{4}{11}}$

Solution (Mass points)

The presence of only ratios in the problem essentially cries out for mass points.

As per the problem, we assign a mass of 1 to point A, and a mass of 3 to D. Then, to balance A and D on T, T has a mass of 4.

Now, were we to assign a mass of 1 to B and a mass of 4 to E, we'd have $5T$. Scaling this down by $4/5$ (to get $4T$, which puts B and E in terms of the masses of A and D), we assign a mass of $\frac{4}{5}$ to B and a mass of $\frac{16}{5}$ to E.

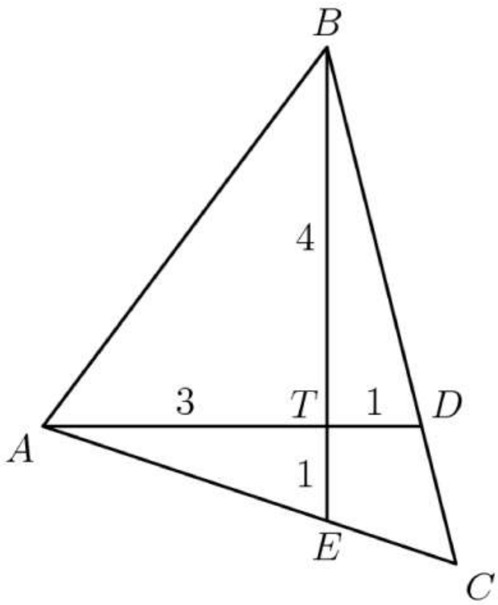
Now, to balance A and C on E, we must give C a mass of $\frac{16}{5} - 1 = \frac{11}{5}$.

Finally, the ratio of CD to BD is given by the ratio of the mass of B to the mass of C, which is $\frac{4}{5} \cdot \frac{5}{11} = \boxed{\textbf{(D)}\ \frac{4}{11}}$.

Solution (Coordinates)

Affine transformations preserve ratios of distances, and for any pair of triangles there is an affine transformation that maps the first one onto the second one. This is why the answer is the same for any $\triangle ABC$, and we just need to compute it for any single triangle.

We can choose the points $A = (-3, 0)$, $B = (0, 4)$, and $D = (1, 0)$. This way we will have $T = (0, 0)$, and $E = (0, -1)$. The situation is shown in the picture below:



The point C is the intersection of the lines BD and AE. The points on the first line have the form $(t, 4 - 4t)$, the points on the second line have the form $(t, -1 - t/3)$. Solving for t we get $t = 15/11$, hence $C = (15/11, -16/11)$.

The ratio CD/BD can now be computed simply by observing the x coordinates of B, C, and D:

$$\frac{CD}{BD} = \frac{15/11 - 1}{1 - 0} = \boxed{\textbf{(D)} \frac{4}{11}}$$

See also

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2004 AMC 10B Problems/Problem 21

Contents

Problem

Let 1; 4; $\ldots$ and 9; 16; $\ldots$ be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

$\mathrm{(A) \ } 3722 \qquad \mathrm{(B) \ } 3732 \qquad \mathrm{(C) \ } 3914 \qquad \mathrm{(D) \ } 3924 \qquad \mathrm{(E) \ } 4007$

Solution

Solution 1

The two sets of terms are $A=\{ 3k+1 : 0\leq k < 2004 \}$ and $B=\{ 7l+9 : 0\leq l<2004\}$.

Now $S=A\cup B$. We can compute $|S|=|A\cup B|=|A|+|B|-|A\cap B|=4008-|A\cap B|$. We will now find $|A\cap B|$.

Consider the numbers in B. We want to find out how many of them lie in A. In other words, we need to find out the number of valid values of l for which $7l+9\in A$.

The fact "$7l+9\in A$" can be rewritten as "$1\leq 7l+9 \leq 3\cdot 2003 + 1$, and $7l+9\equiv 1 \pmod 3$".

The first condition gives $0\leq l\leq 857$, the second one gives $l\equiv 1 \pmod 3$.

Thus the good values of l are $\{1,4,7,\dots,856\}$, and their count is $858/3=286$.

Therefore $|A\cap B|=286$, and thus $|S|=4008-|A\cap B|=\boxed{3722}$.

Solution 2

Shift down the first sequence by 1 and the second by 9 so that the two sequences become $0,3,6,\cdots,6009$ and $0,7,14,\cdots,14028$. The first becomes multiples of 3 and the second becomes multiples of 7. Their intersection is the multiples of 21 up to 6009. There are $\lfloor\frac{6009}{21}\rfloor=286$ multiples of 21. There are $4008-286=\boxed{\mathrm{(A)}\ 3722}$ distinct numbers in S.

See also

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2004 AMC 10B Problems/Problem 22

Problem

A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

$\text{(A)}\ \frac{3\sqrt{5}}{2} \qquad \text{(B)}\ \frac{7}{2} \qquad \text{(C)}\ \sqrt{15} \qquad \text{(D)}\ \frac{\sqrt{65}}{2} \qquad \text{(E)}\ \frac{9}{2}$

Solution

This is obviously a right triangle. Pick a coordinate system so that the right angle is at $(0,0)$ and the other two vertices are at $(12,0)$ and $(0,5)$.

As this is a right triangle, the center of the circumcircle is in the middle of the hypotenuse, at $(6,2.5)$.

The radius r of the inscribed circle can be computed using the well-known identity $\frac{rP}{2}=S$, where S is the area of the triangle and P its perimeter. In our case, $S=5\cdot 12/2=30$ and $P=5+12+13=30$, thus $r=2$. As the inscribed circle touches both legs, its center must be at $(r,r)=(2,2)$.

The distance of these two points is then $\sqrt{(6-2)^2+(2.5-2)^2}=\sqrt{16.25}=\sqrt{\frac{65}{4}}=\boxed{\frac{\sqrt{65}}{2}}$.

See also

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2004 AMC 10B Problems/Problem 23

Contents

Problem

Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

$(\mathrm{A})\ \frac{1}{4} \qquad (\mathrm{B})\ \frac{5}{16} \qquad (\mathrm{C})\ \frac{3}{8} \qquad (\mathrm{D})\ \frac{7}{16} \qquad (\mathrm{E})\ \frac{1}{2}$

Solution

If the orientation of the cube is fixed, there are $2^6 = 64$ possible arrangements of colors on the faces. There are

$$2\binom{6}{6} = 2$$

arrangements in which all six faces are the same color and

$$2\binom{6}{5} = 12$$

arrangements in which exactly five faces have the same color. In each of these cases the cube can be placed so that the four vertical faces have the same color. The only other suitable arrangements have four faces of one color, with the other color on a pair of opposing faces. Since there are three pairs of opposing faces, there are $2(3) = 6$ such arrangements. The total number of suitable arrangements is therefore $2 + 12 + 6 = 20$, and the probability is $\frac{20}{64} = \frac{5}{16}$.

Solution 2

Label the six sides of the cube by numbers 1 to 6 as on a classic dice. Then the "four vertical faces" can be: $\{1, 2, 5, 6\}$, $\{1, 3, 4, 6\}$, or $\{2, 3, 4, 5\}$.

Let A be the set of colorings where $1, 2, 5, 6$ are all of the same color, similarly let B and C be the sets of good colorings for the other two sets of faces.

There are $2^6 = 64$ possible colorings, and there are $|A \cup B \cup C|$ good colorings. Thus the result is $\frac{|A \cup B \cup C|}{64}$. We need to compute $|A \cup B \cup C|$.

Using the Principle of Inclusion-Exclusion we can write

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Clearly $|A| = |B| = |C| = 2^3 = 8$, as we have two possibilities for the common color of the four vertical faces, and two possibilities for each of the horizontal faces.

What is $A \cap B$? The faces $1, 2, 5, 6$ must have the same color, and at the same time faces $1, 3, 4, 6$ must have the same color. It turns out that $A \cap B = A \cap C = B \cap C = A \cap B \cap C =$ the set containing just the two cubes where all six faces have the same color.

Therefore $|A \cup B \cup C| = 8 + 8 + 8 - 2 - 2 - 2 + 2 = 20$, and the result is $\frac{20}{64} = \boxed{\frac{5}{16}}$.

Solution 3

Suppose we break the situation into cases that contain four vertical faces of the same color:

I. Two opposite sides of same color: There are 3 ways to choose the two sides, and then two colors possible, so $3 \cdot 2 = 6$.

II. One face different from all the others: There are 6 ways to choose this face, and 2 colors, so $6 \cdot 2 = 12$.

III. All faces are the same: There are 2 colors, and so two ways for all faces to be the same.

Adding them up, we have a total of 20 ways to have four vertical faces the same color. The are 2^6 ways to color the cube, so the answer is $\frac{20}{64} = \boxed{\frac{5}{16}}$.

See also

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2004 AMC 10B Problems/Problem 24

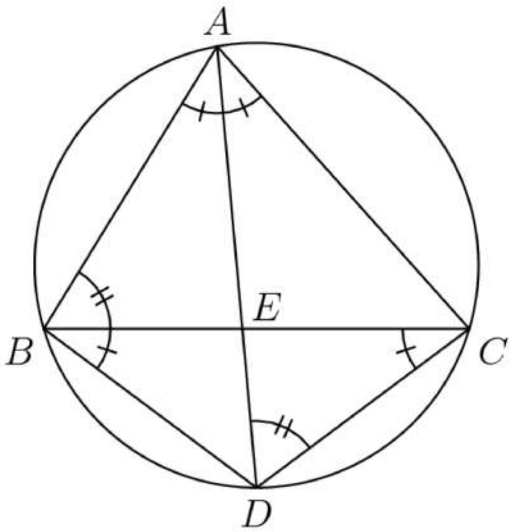
In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of AD/CD?

$$\text{(A) } \frac{9}{8} \quad \text{(B) } \frac{5}{3} \quad \text{(C) } 2 \quad \text{(D) } \frac{17}{7} \quad \text{(E) } \frac{5}{2}$$

Solution

Set $\overline{BD}$'s length as x. CD's length must also be x since $\angle BAD$ and $\angle DAC$ intercept arcs of equal length(because $\angle BAD = \angle DAC$). Using Ptolemy's Theorem, $7x + 8x = 9(AD)$. The ratio is $\boxed{\frac{5}{3}} \implies (B)$

Solution 2



Let $E = \overline{BC} \cap \overline{AD}$. Observe that $\angle ABC = \angle ADC$ because they subtend the same arc. Furthermore, $\angle BAP = \angle EAC$, so $\triangle ABE$ is similar to $\triangle ADC$ by AAA similarity. Then $\frac{AD}{AB} = \frac{CD}{BE}$. By angle bisector theorem, $\frac{7}{BE} = \frac{8}{CE}$ so $\frac{7}{BE} = \frac{8}{9 - BE}$ which gives $BE = \frac{21}{5}$. Plugging this into the similarity proportion gives: $\frac{AD}{7} = \frac{CD}{\frac{21}{5}} \implies \frac{AD}{CD} = \boxed{\frac{5}{3}} = \textbf{B}$.

See Also

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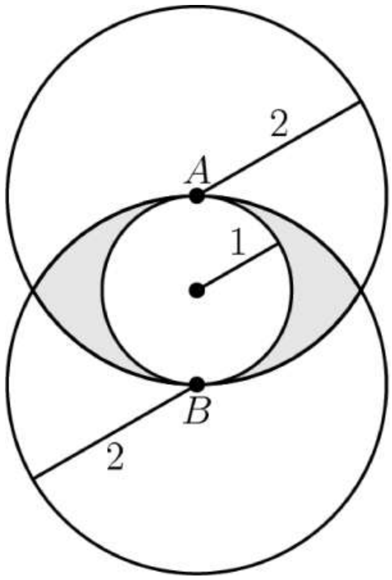
2004 AMC 10B Problems/Problem 25

Problem

A circle of radius 1 is internally tangent to two circles of radius 2 at points A and B, where AB is a diameter of the smaller circle. What is the area of the region, shaded in the picture, that is outside the smaller circle and inside each of the two larger circles?

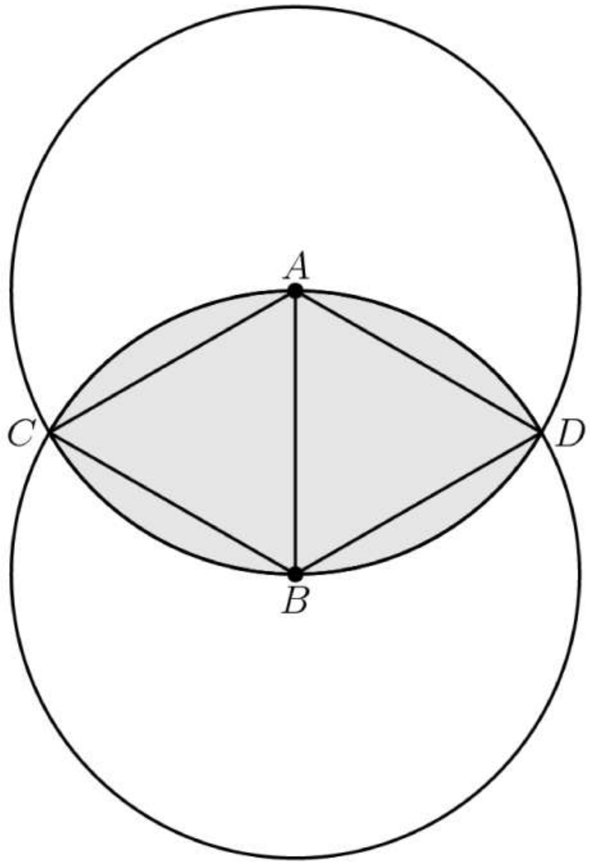
$\mathrm{(A)}\ \frac{5}{3}\pi - 3\sqrt{2} \qquad \mathrm{(B)}\ \frac{5}{3}\pi - 2\sqrt{3} \qquad \mathrm{(C)}\ \frac{8}{3}\pi - 3\sqrt{3} \qquad \mathrm{(D)}\ \frac{8}{3}\pi - 3\sqrt{2} \qquad \mathrm{(E)}\ \frac{8}{3}\pi - 2\sqrt{3}$



Solution

The area of the small circle is π. We can add it to the shaded region, compute the area of the new region, and then subtract the area of the small circle from the result.

Let C and D be the intersections of the two large circles. Connect them to A and B to get the picture below:



Now obviously the triangles $\triangle ABC$ and $\triangle ABD$ are equilateral with side 2.

Take a look at the bottom circle. The angle ABC is 60°, hence the sector ABC is $1/6$ of the circle. The same is true for the sector ABD of the bottom circle, and sectors CAB and BAD of the top circle.

If we now sum the areas of these four sectors, we will almost get the area of the new shaded region - except that each of the two equilateral triangles will be counted twice. These triangles have a base of 2 and a height of $\sqrt{3}$.

Hence the area of the new shaded region is $4\cdot\left(\frac{1}{6}\cdot\pi\cdot 2^2\right) - 2\cdot\left(\frac{1}{2}\cdot 2\cdot\sqrt{3}\right) = \frac{8}{3}\pi - 2\sqrt{3}$, and the area of the original shared region is $\left(\frac{8}{3}\pi - 2\sqrt{3}\right) - \pi = \boxed{\mathrm{(B)}\ \frac{5}{3}\pi - 2\sqrt{3}}$.

See also

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