2005 AMC 10A Problems/Problem 1

Problem

While eating out, Mike and Joe each tipped their server 2 dollars. Mike tipped 10% of his bill and Joe tipped 20% of his bill. What was the difference, in dollars between their bills?

$\mathrm{(A)}\ 2\qquad\mathrm{(B)}\ 4\qquad\mathrm{(C)}\ 5\qquad\mathrm{(D)}\ 10\qquad\mathrm{(E)}\ 20$

Solution

Let m be Mike's bill and j be Joe's bill.

$\frac{10}{100}m=2$

$m=20$

$\frac{20}{100}j=2$

$j=10$

So the desired difference is $m-j=20-10=10 \Rightarrow D$

See Also

- 2005 AMC 10A Problems
- Next Problem

Category: Introductory Algebra Problems

2005 AMC 10A Problems/Problem 2

Problem

For each pair of real numbers $a \neq b$, define the operation $\star$ as

$$(a \star b) = \frac{a+b}{a-b}.$$

What is the value of $((1 \star 2) \star 3)$?

$$\text{(A) } -\frac{2}{3} \qquad \text{(B) } -\frac{1}{5} \qquad \text{(C) } 0 \qquad \text{(D) } \frac{1}{2} \qquad \text{(E) This value is not defined.}$$

Solution

$$((1 \star 2) \star 3) = \left(\left(\frac{1+2}{1-2}\right) \star 3\right) = (-3 \star 3) = \frac{-3+3}{-3-3} = 0 \Longrightarrow \text{(C)}$$

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 2

The following problem is from both the 2005 AMC 12A #2 and 2005 AMC 10A #3, so both problems redirect to this page.

Problem

The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution. What is the value of b?

$\mathrm{(A)}\ -8 \qquad \mathrm{(B)}\ -4 \qquad \mathrm{(C)}\ 2 \qquad \mathrm{(D)}\ 4 \qquad \mathrm{(E)}\ 8$

Solution

$$2x + 7 = 3 \Longrightarrow x = -2, \quad -2b - 10 = -2 \Longrightarrow -2b = 8 \Longrightarrow b = -4\ \mathrm{(B)}$$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 1	Followed by Problem 3
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2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2005 AMC 10A Problems/Problem 4

Problem

A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle?

$\mathrm{(A) \ } \frac{1}{4}x^2\qquad \mathrm{(B) \ } \frac{2}{5}x^2\qquad \mathrm{(C) \ } \frac{1}{2}x^2\qquad \mathrm{(D) \ } x^2\qquad \mathrm{(E) \ } \frac{3}{2}x^2$

Solution

Let the width of the rectangle be w. Then the length is $2w$

Using the Pythagorean Theorem:

$x^2 = w^2 + (2w)^2$

$x^2 = 5w^2$

The area of the rectangle is $2w^2 = \frac{2}{5}x^2$

(B)

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2005_AMC_10A_Problems/Problem_4&oldid=80615"

Category: Introductory Geometry Problems

2005 AMC 10A Problems/Problem 5

Problem

A store normally sells windows at 100 each. This week the store is offering one free window for each purchase of four. Dave needs seven windows and Doug needs eight windows. How many dollars will they save if they purchase the windows together rather than separately?

$\mathrm{(A) \ } 100\qquad \mathrm{(B) \ } 200\qquad \mathrm{(C) \ } 300\qquad \mathrm{(D) \ } 400\qquad \mathrm{(E) \ } 500$

Solution

The store's offer means that every 5th window is free.

Dave would get $\lfloor\frac{7}{5}\rfloor = 1$ free window.

Doug would get $\lfloor\frac{8}{5}\rfloor = 1$ free window.

This is a total of 2 free windows.

Together, they would get $\lfloor\frac{8+7}{5}\rfloor = \lfloor\frac{15}{5}\rfloor = 3$ free windows.

So they get $3-2=1$ additional window if they purchase the windows together.

Therefore they save $1\cdot100=100\Rightarrow A$

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2005_AMC_10A_Problems/Problem_5&oldid=54263"

Category: Introductory Algebra Problems

2005 AMC 10A Problems/Problem 6

Problem

The average (mean) of 20 numbers is 30, and the average of 30 other numbers is 20. What is the average of all 50 numbers?

$\mathrm{(A)}\ 23 \qquad \mathrm{(B)}\ 24 \qquad \mathrm{(C)}\ 25 \qquad \mathrm{(D)}\ 26 \qquad \mathrm{(E)}\ 27$

Solution

Since the average of the first 20 numbers is 30, their sum is $20\cdot30=600$.

Since the average of 30 other numbers is 20, their sum is $30\cdot20=600$.

So the sum of all 50 numbers is $600+600=1200$

Therefore, the average of all 50 numbers is $\frac{1200}{50}=24\Longrightarrow \mathrm{(B)}$

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

Category: Introductory Algebra Problems

2005 AMC 10A Problems/Problem 7

Problem

Josh and Mike live 13 miles apart. Yesterday Josh started to ride his bicycle toward Mike's house. A little later Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met?

$(\mathrm{A})\ 4 \qquad (\mathrm{B})\ 5 \qquad (\mathrm{C})\ 6 \qquad (\mathrm{D})\ 7 \qquad (\mathrm{E})\ 8$

Solution

Let m be the distance in miles that Mike rode.

Since Josh rode for twice the length of time as Mike and at four-fifths of Mike's rate, he rode $2\cdot\frac{4}{5}\cdot m=\frac{8}{5}m$ miles.

Since their combined distance was 13 miles,

$\frac{8}{5}m+m=13$

$\frac{13}{5}m=13$

$m=5\Longrightarrow (\mathrm{B})$

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

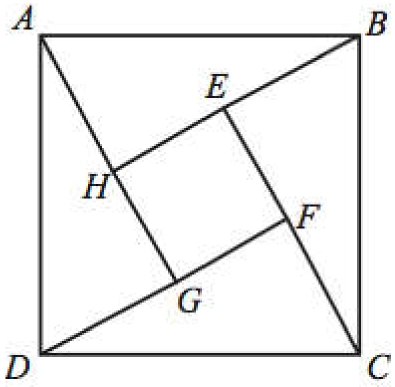
Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2005_AMC_10A_Problems/Problem_7&oldid=54265"

Category: Introductory Algebra Problems

2005 AMC 10A Problems/Problem 8

Problem

In the figure, the length of side AB of square $ABCD$ is $\sqrt{50}$ and BE=1. What is the area of the inner square $EFGH$?



$\mathrm{(A) \ } 25\qquad \mathrm{(B) \ } 32\qquad \mathrm{(C) \ } 36\qquad \mathrm{(D) \ } 40\qquad \mathrm{(E) \ } 42$

Solution

We see that side BE, which we know is 1, is also the shorter leg of one of the four right triangles (which are congruent, I'll not prove this). So, $AH = 1$. Then $HB = HE + BE = HE + 1$, and HE is one of the sides of the square whose area we want to find. So:

$$1^2 + (HE + 1)^2 = \sqrt{50}^2$$

$$1 + (HE + 1)^2 = 50$$

$$(HE + 1)^2 = 49$$

$$HE + 1 = 7$$

$$HE = 6$$

So, the area of the square is $6^2 = \boxed{36} \Rightarrow \mathrm{(C)}$.

See Also

Categories: Introductory Geometry Problems | Area Problems

2005 AMC 10A Problems/Problem 9

Problem

Three tiles are marked X and two other tiles are marked O. The five tiles are randomly arranged in a row. What is the probability that the arrangement reads $XOXOX$?

$\mathrm{(A)}\ \frac{1}{12}\qquad \mathrm{(B)}\ \frac{1}{10}\qquad \mathrm{(C)}\ \frac{1}{6}\qquad \mathrm{(D)}\ \frac{1}{4}\qquad \mathrm{(E)}\ \frac{1}{3}$

Solution

There are $\frac{5!}{2!3!}=10$ distinct arrangements of three X's and two O's.

There is only 1 distinct arrangement that reads $XOXOX$

Therefore the desired probability is $\boxed{\frac{1}{10}} \Rightarrow \mathrm{(B)}$

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem
- Combination

Category: Introductory Combinatorics Problems

2005 AMC 10A Problems/Problem 10

Contents

Problem

There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

$\mathrm{(A)}\ -16\qquad\mathrm{(B)}\ -8\qquad\mathrm{(C)}\ 0\qquad\mathrm{(D)}\ 8\qquad\mathrm{(E)}\ 20$

Solution

A quadratic equation has exactly one root if and only if it is a perfect square. So set

$4x^2 + ax + 8x + 9 = (mx+n)^2$

$4x^2 + ax + 8x + 9 = m^2x^2 + 2mnx + n^2$

Two polynomials are equal only if their coefficients are equal, so we must have

$m^2 = 4, n^2 = 9$

$m = \pm 2, n = \pm 3$

$a + 8 = 2mn = \pm 2 \cdot 2 \cdot 3 = \pm 12$

$a = 4$ or $a = -20$.

So the desired sum is $(4)+(-20)=-16 \Longrightarrow \mathrm{(A)}$

Alternatively, note that whatever the two values of a are, they must lead to equations of the form $px^2+qx+r=0$ and $px^2-qx+r=0$. So the two choices of a must make $a_1+8=q$ and $a_2+8=-q$ so $a_1+a_2+16=0$ and $a_1+a_2=-16 \Longrightarrow \mathrm{(A)}$.

Alternate Solution

Since this quadratic must have a double root, the discriminant of the quadratic formula for this quadratic must be 0. Therefore, we must have

$$(a+8)^2-4(4)(9)=0 \implies a^2+16a-80.$$

We can use the quadratic formula to solve for its roots (we can ignore the things in the radical sign as they will cancel out due to the $\pm$ sign when added). So we must have

$$\frac{-16+\sqrt{\text{something}}}{2}+\frac{-16-\sqrt{\text{something}}}{2}.$$

Therefore, we have $(-16)(2)/2=-16 \implies \boxed{A}$.

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

2005 AMC 10A Problems/Problem 11

Problem

A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

$\mathrm{(A)}\ 3\qquad \mathrm{(B)}\ 4\qquad \mathrm{(C)}\ 5\qquad \mathrm{(D)}\ 6\qquad \mathrm{(E)}\ 7$

Solution

Since there are n^2 little faces on each face of the big wooden cube, there are $6n^2$ little faces painted red.

Since each unit cube has 6 faces, there are $6n^3$ little faces total.

Since one-fourth of the little faces are painted red,

$$\frac{6n^2}{6n^3}=\frac{1}{4}$$

$$\frac{1}{n}=\frac{1}{4}$$

$$n=4 \Longrightarrow \mathrm{(B)}$$

See Also

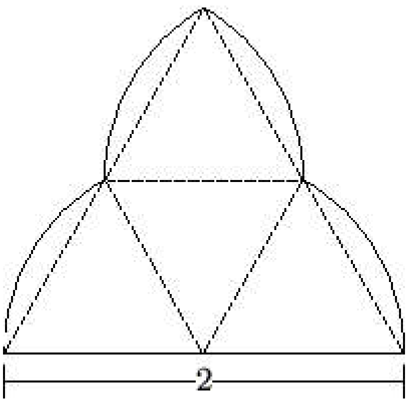
- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

Category: Introductory Geometry Problems

2005 AMC 10A Problems/Problem 12

Problem

The figure shown is called a trefoil and is constructed by drawing circular sectors about the sides of the congruent equilateral triangles. What is the area of a trefoil whose horizontal base has length 2?



$\text{(A)}\ \frac{1}{3}\pi + \frac{\sqrt{3}}{2} \qquad \text{(B)}\ \frac{2}{3}\pi \qquad \text{(C)}\ \frac{2}{3}\pi + \frac{\sqrt{3}}{4} \qquad \text{(D)}\ \frac{2}{3}\pi + \frac{\sqrt{3}}{3} \qquad \text{(E)}\ \frac{2}{3}\pi + \frac{\sqrt{3}}{2}$

Solution

The area of the trefoil is equal to the area of a small equilateral triangle plus the area of four 60° sectors with a radius of $\frac{2}{2} = 1$ minus the area of a small equilateral triangle.

This is equivalent to the area of four 60° sectors with a radius of 1.

So the answer is:

$$4 \cdot \frac{60}{360} \cdot \pi \cdot 1^2 = \frac{4}{6} \cdot \pi = \frac{2}{3}\pi \Rightarrow B$$

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

Categories: Introductory Geometry Problems | Area Problems

2005 AMC 10A Problems/Problem 13

Problem

How many positive integers n satisfy the following condition:

$(130n)^{50} > n^{100} > 2^{200}$?

$(\mathrm{A})\ 0 \qquad (\mathrm{B})\ 7 \qquad (\mathrm{C})\ 12 \qquad (\mathrm{D})\ 65 \qquad (\mathrm{E})\ 125$

Solution

We're given $(130n)^{50} > n^{100} > 2^{200}$, so

$\sqrt[50]{(130n)^{50}} > \sqrt[50]{n^{100}} > \sqrt[50]{2^{200}}$ (because all terms are positive) and thus

$130n > n^2 > 2^4$

$130n > n^2 > 16$

Solving each part separately:

$n^2 > 16 \Longrightarrow n > 4$

$130n > n^2 \Longrightarrow 130 > n$

So $4 < n < 130$.

Therefore the answer is the number of positive integers over the interval $(4, 130)$ which is $125 \Longrightarrow (\mathrm{E})$.

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

Category: Introductory Algebra Problems

2005 AMC 10A Problems/Problem 14

Contents

Problem

How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

$(\mathrm{A})\ 41 \qquad (\mathrm{B})\ 42 \qquad (\mathrm{C})\ 43 \qquad (\mathrm{D})\ 44 \qquad (\mathrm{E})\ 45$

Solution

If the middle digit is the average of the first and last digits, twice the middle digit must be equal to the sum of the first and last digits.

Doing some casework:

If the middle digit is 1, possible numbers range from 111 to 210. So there are 2 numbers in this case.

If the middle digit is 2, possible numbers range from 123 to 420. So there are 4 numbers in this case.

If the middle digit is 3, possible numbers range from 135 to 630. So there are 6 numbers in this case.

If the middle digit is 4, possible numbers range from 147 to 840. So there are 8 numbers in this case.

If the middle digit is 5, possible numbers range from 159 to 951. So there are 9 numbers in this case.

If the middle digit is 6, possible numbers range from 369 to 963. So there are 7 numbers in this case.

If the middle digit is 7, possible numbers range from 579 to 975. So there are 5 numbers in this case.

If the middle digit is 8, possible numbers range from 789 to 987. So there are 3 numbers in this case.

If the middle digit is 9, the only possible number is 999. So there is 1 number in this case.

So the total number of three-digit numbers that satisfy the property is
$2+4+6+8+9+7+5+3+1=45 \Rightarrow E$

Solution 2 (much faster and slicker)

Alternatively, we could note that the middle digit is uniquely defined by the first and third digits, since it is half of their sum. This also means that the sum of the first and third digits must be even. Since even numbers are formed either by adding two odd numbers or two even numbers, we can split our problem into 2 cases:

If both the first digit and the last digit are odd, then we have 1, 3, 5, 7, or 9 as choices for each of these digits, and there are $5 \cdot 5 = 25$ numbers in this case.

If both the first and last digits are even, then we have 2, 4, 6, 8 as our choices for the first digit and 0, 2, 4, 6, 8 for the third digit. There are $4 \cdot 5 = 20$ numbers here.

The total number, then, is $20+25=45 \Rightarrow E$

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem

2005 AMC 10A Problems/Problem 15

Contents

Problem

How many positive cubes divide $3! \cdot 5! \cdot 7!$?

$\mathrm{(A) \ } 2\qquad \mathrm{(B) \ } 3\qquad \mathrm{(C) \ } 4\qquad \mathrm{(D) \ } 5\qquad \mathrm{(E) \ } 6$

Solution

Solution 1

$3! \cdot 5! \cdot 7! = (3\cdot2\cdot1) \cdot (5\cdot4\cdot3\cdot2\cdot1) \cdot (7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1) = 2^{8}\cdot3^{4}\cdot5^{2}\cdot7^{1}$

Therefore, a perfect cube that divides $3! \cdot 5! \cdot 7!$ must be in the form $2^{a}\cdot3^{b}\cdot5^{c}\cdot7^{d}$ where a, b, c, and d are nonnegative multiples of 3 that are less than or equal to 8, 4, 2 and 1, respectively.

So:

$a\in\{0,3,6\}$ (3 possibilities)

$b\in\{0,3\}$ (2 possibilities)

$c\in\{0\}$ (1 possibility)

$d\in\{0\}$(1 possibility)

So the number of perfect cubes that divide $3! \cdot 5! \cdot 7!$ is $3\cdot2\cdot1\cdot1 = 6 \Rightarrow \mathrm{(E)}$

Solution 2

If you factor $3! \cdot 5! \cdot 7!$ You get

$2^7 \cdot 3^4 \cdot 5^2$

There are 3 ways for the first factor of a cube: 2^0, 2^3, and 2^6. And the second ways are: 3^0, and 3^3.

$3 \cdot 2 = 6$ Answer : $\boxed{E}$

See Also

- 2005 AMC 10A Problems
- Previous Problem
- Next Problem
- Factorial
- Prime factorization

2005 AMC 10A Problems/Problem 16

Problem

The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property?

$\mathrm{(A)}\ 5\qquad\mathrm{(B)}\ 7\qquad\mathrm{(C)}\ 9\qquad\mathrm{(D)}\ 10\qquad\mathrm{(E)}\ 19$

Solution

Let the number be $10a+b$ where a and b are the tens and units digits of the number.

So $(10a+b)-(a+b)=9a$ must have a units digit of 6

This is only possible if $9a=36$, so $a=4$ is the only way this can be true.

So the numbers that have this property are $40, 41, 42, 43, 44, 45, 46, 47, 48, 49$.

Therefore the answer is $10 \Rightarrow D$

See Also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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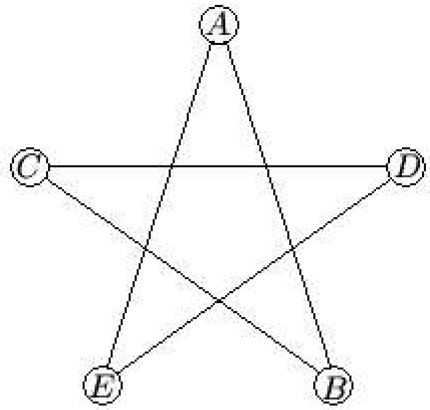
All AMC 10 Problems and Solutions	

Categories: Introductory Geometry Problems | Area Ratio Problems

2005 AMC 10A Problems/Problem 17

Problem

In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments AB, BC, CD, DE, and EA form an arithmetic sequence, although not necessarily in that order. What is the middle term of the arithmetic sequence?



$\mathrm{(A) \ } 9\qquad \mathrm{(B) \ } 10\qquad \mathrm{(C) \ } 11\qquad \mathrm{(D) \ } 12\qquad \mathrm{(E) \ } 13$

Solution

Each corner (a,b,c,d,e) goes to two sides/numbers. (A goes to AE and AB, D goes to DC and DE). The sum of every term is equal to $2(3+5+6+7+9)=60$

Since the middle term in an arithmetic sequence is the average of all the terms in the sequence, the middle number is $\frac{60}{5}=12 \Rightarrow D$

See Also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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Categories: Introductory Geometry Problems | Area Ratio Problems

2005 AMC 10A Problems/Problem 18

Problem

Team A and team B play a series. The first team to win three games wins the series. Each team is equally likely to win each game, there are no ties, and the outcomes of the individual games are independent. If team B wins the second game and team A wins the series, what is the probability that team B wins the first game?

$\mathrm{(A)}\ \frac{1}{5}\qquad\mathrm{(B)}\ \frac{1}{4}\qquad\mathrm{(C)}\ \frac{1}{3}\qquad\mathrm{(D)}\ \frac{1}{2}\qquad\mathrm{(E)}\ \frac{2}{3}$

Solution

There are at most 5 games played.

If team B won the first two games, team A would need to win the next three games. So the only possible order of wins is BBAAA.

If team A won the first game, and team B won the second game, the possible order of wins are: ABBAA, ABABA, and ABAAX, where X denotes that the 5th game wasn't played.

Since ABAAX is dependent on the outcome of 4 games instead of 5, it is twice as likely to occur and can be treated as two possibilities.

Since there is 1 possibility where team B wins the first game and 5 total possibilities, the desired probability is $\frac{1}{5} \Rightarrow A$

See Also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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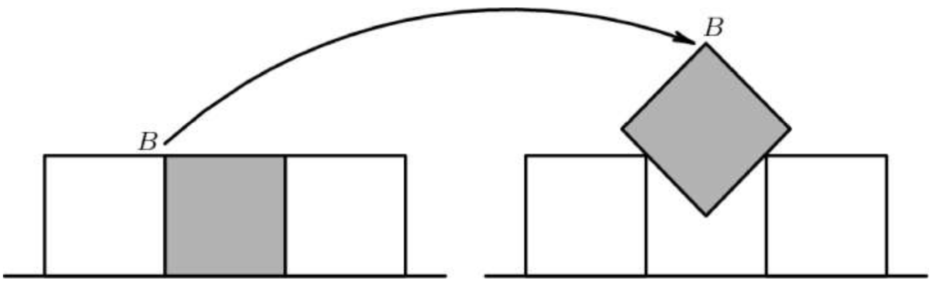
All AMC 10 Problems and Solutions	

Categories: Introductory Geometry Problems | Area Ratio Problems

2005 AMC 10A Problems/Problem 19

Problem

Three one-inch squares are placed with their bases on a line. The center square is lifted out and rotated 45 degrees, as shown. Then it is centered and lowered into its original location until it touches both of the adjoining squares. How many inches is the point B from the line on which the bases of the original squares were placed?

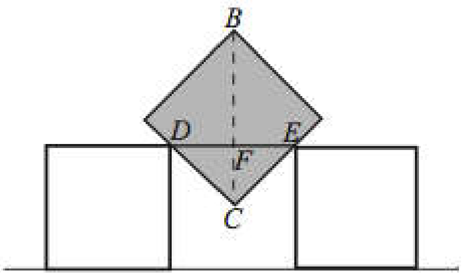


$\textbf{(A)}\ 1\qquad\textbf{(B)}\ \sqrt{2}\qquad\textbf{(C)}\ \frac{3}{2}\qquad\textbf{(D)}\ \sqrt{2}+\frac{1}{2}\qquad\textbf{(E)}\ 2$

Solution

Consider the rotated middle square shown in the figure. It will drop until length DE is 1 inch. Then, because DEC is a $45^\circ - 45^\circ - 90^\circ$ triangle, $EC = \frac{\sqrt{2}}{2}$, and $FC = \frac{1}{2}$. We know that $BC = \sqrt{2}$, so the distance from B to the line is

$$BC - FC + 1 = \sqrt{2} - \frac{1}{2} + 1 = \sqrt{2} + \frac{1}{2}\ (D).$$



See Also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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All AMC 10 Problems and Solutions	

2005 AMC 10A Problems/Problem 20

Problem

An equiangular octagon has four sides of length 1 and four sides of length $\frac{\sqrt{2}}{2}$, arranged so that no two consecutive sides have the same length. What is the area of the octagon?

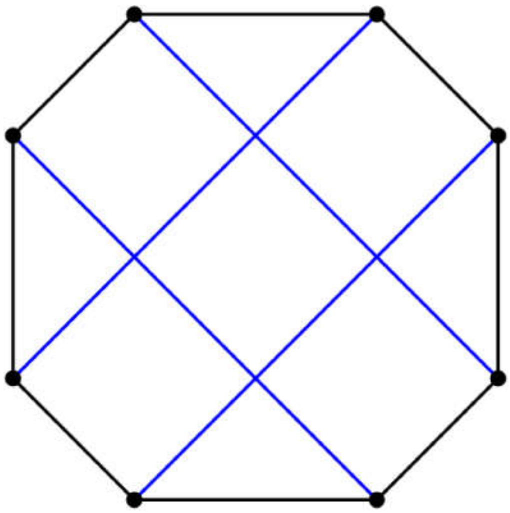
$\mathrm{(A)}\ \frac{7}{2}\qquad\mathrm{(B)}\ \frac{7\sqrt{2}}{2}\qquad\mathrm{(C)}\ \frac{5+4\sqrt{2}}{2}\qquad\mathrm{(D)}\ \frac{4+5\sqrt{2}}{2}\qquad\mathrm{(E)}\ 7$

Solution

The area of the octagon can be divided up into 5 squares with side $\frac{\sqrt{2}}{2}$ and 4 right triangles, which are half the area of each of the squares.

Therefore, the area of the octagon is equal to the area of $5+4\left(\frac{1}{2}\right)=7$ squares.

The area of each square is $\left(\frac{\sqrt{2}}{2}\right)^2=\frac{1}{2}$, so the area of 7 squares is $\frac{7}{2}\Rightarrow\mathrm{(A)}$.



See Also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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All AMC 10 Problems and Solutions	

2005 AMC 10A Problems/Problem 21

Problem

For how many positive integers n does $1+2+...+n$ evenly divide from $6n$?

$\mathrm{(A) \ } 3\qquad \mathrm{(B) \ } 5\qquad \mathrm{(C) \ } 7\qquad \mathrm{(D) \ } 9\qquad \mathrm{(E) \ } 11$

Solution

If $1+2+...+n$ evenly divides $6n$, then $\frac{6n}{1+2+...+n}$ is an integer.

Since $1+2+...+n = \frac{n(n+1)}{2}$ we may substitute the RHS in the above fraction. So the problem asks us for how many positive integers n is $\frac{6n}{\frac{n(n+1)}{2}} = \frac{12}{n+1}$ an integer, or equivalently when $k(n+1) = 12$ for a positive integer k.

$\frac{12}{n+1}$ is an integer when $n+1$ is a factor of 12.

The factors of 12 are 1, 2, 3, 4, 6, and 12, so the possible values of n are 0, 1, 2, 3, 5, and 11.

But 0 isn't a positive integer, so only 1, 2, 3, 5, and 11 are possible values of n. Therefore the number of possible values of n is $5 \Longrightarrow \boxed{\mathrm{(B)}}$.

See also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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2005 AMC 10A Problems/Problem 22

Problem

Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T?

$\mathrm{(A) \ } 166\qquad \mathrm{(B) \ } 333\qquad \mathrm{(C) \ } 500\qquad \mathrm{(D) \ } 668\qquad \mathrm{(E) \ } 1001$

Solution

Since the least common multiple $\mathrm{lcm}(4,6)=12$, the elements that are common to S and T must be multiples of 12.

Since $4\cdot2005=8020$ and $6\cdot2005=12030$, several multiples of 12 that are in T won't be in S, but all multiples of 12 that are in S will be in T. So we just need to find the number of multiples of 12 that are in S.

Since $4\cdot3=12$ every 3rd element of S will be a multiple of 12

Therefore the answer is $\lfloor\frac{2005}{3}\rfloor=668 \Longrightarrow \mathrm{(D)}$

See Also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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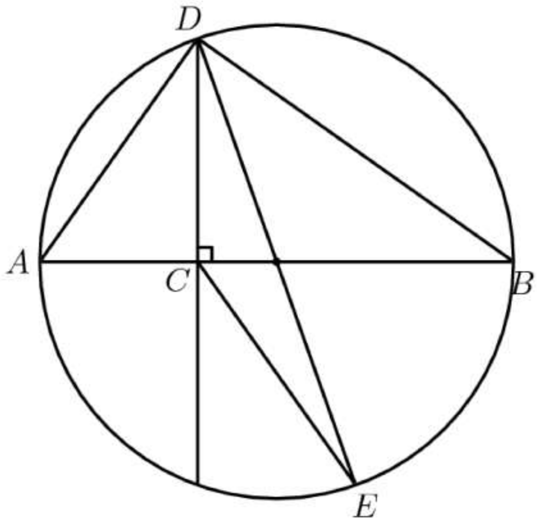
2005 AMC 10A Problems/Problem 23

Contents

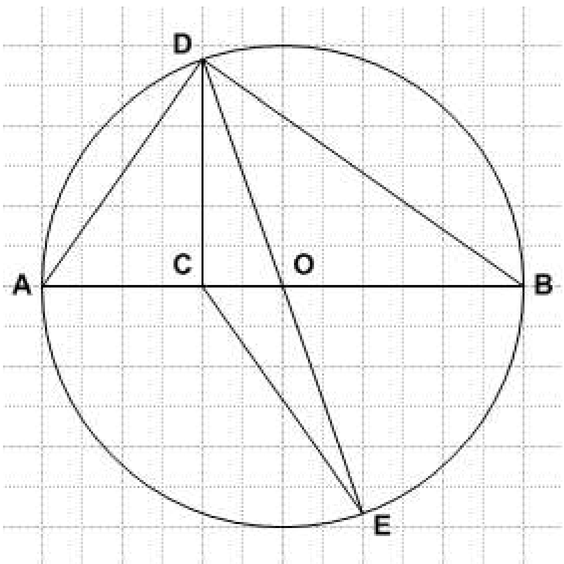
Problem

Let AB be a diameter of a circle and let C be a point on AB with $2 \cdot AC = BC$. Let D and E be points on the circle such that $DC \perp AB$ and DE is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



$\mathrm{(A)}\ \frac{1}{6} \qquad \mathrm{(B)}\ \frac{1}{4} \qquad \mathrm{(C)}\ \frac{1}{3} \qquad \mathrm{(D)}\ \frac{1}{2} \qquad \mathrm{(E)}\ \frac{2}{3}$

Solution 1



Let us assume that the diameter is of length 1.

AC is $\frac{1}{3}$ of diameter and CO is $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$.

OD is the radius of the circle, so using the Pythagorean theorem height CD of $\triangle AOC$ is $\sqrt{(\frac{1}{2})^2-(\frac{1}{6})^2}=\frac{\sqrt{2}}{3}$. This is also the height of the $\triangle ABD$.

Area of the $\triangle DCO$ is $\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{\sqrt{2}}{3}$ = $\frac{\sqrt{2}}{36}$.

The height of $\triangle DCE$ can be found using the area of $\triangle DCO$ and DO as base.

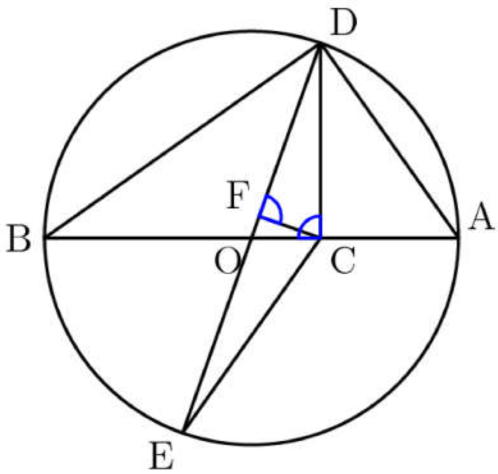
Hence the height of $\triangle DCE$ is $\frac{\frac{\sqrt{2}}{36}}{\frac{1}{2}\cdot\frac{1}{2}}$ = $\frac{\sqrt{2}}{9}$.

The diameter is the base for both the triangles $\triangle DCE$ and $\triangle ABD$.

Hence, the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$ is $\frac{\frac{\sqrt{2}}{9}}{\frac{\sqrt{2}}{3}}$ = $\frac{1}{3}\Rightarrow C$

Solution 2

Since $\triangle DCE$ and $\triangle ABD$ share a base, the ratio of their areas is the ratio of their altitudes. Draw the altitude from C to DE.



$OD=r, OC=\frac{1}{3}r$.

Since $m\angle DCO=m\angle DFC=90^\circ$, then $\triangle DCO\cong\triangle DFC$. So the ratio of the two altitudes is $\frac{CF}{DC}=\frac{OC}{DO}=\frac{1}{3}\Rightarrow\text{(C)}$

Solution 3

Say the center of the circle is point O; Without loss of generality, assume $AC=2$, so $CB=4$ and the diameter and radius are 6 and 3, respectively. Therefore, $CO=1$, and $DO=3$. The area of $\triangle DCE$ can be expressed as $\frac{1}{2}(CD)(6)\sin\,(CDE)$. $\frac{1}{2}(CD)(6)$ happens to be the area of $\triangle ABD$. Furthermore, $\sin\,CDE=\frac{CO}{DO}$, or $\frac{1}{3}$. Therefore, the ratio is $\frac{1}{3}$.

See also

2005 AMC 10A Problems/Problem 24

Problem

For each positive integer $n > 1$, let $P(n)$ denote the greatest prime factor of n. For how many positive integers n is it true that both $P(n) = \sqrt{n}$ and $P(n+48) = \sqrt{n+48}$?

$\mathrm{(A) \ } 0 \qquad \mathrm{(B) \ } 1 \qquad \mathrm{(C) \ } 3 \qquad \mathrm{(D) \ } 4 \qquad \mathrm{(E) \ } 5$

Solution

If $P(n) = \sqrt{n}$, then $n = p_1^2$, where p_1 is a prime number.

If $P(n+48) = \sqrt{n+48}$, then $n + 48 = p_2^2$, where p_2 is a different prime number.

So:

$p_2^2 = n + 48$

$p_1^2 = n$

$p_2^2 - p_1^2 = 48$

$(p_2 + p_1)(p_2 - p_1) = 48$

Since $p_1 > 0$: $(p_2 + p_1) > (p_2 - p_1)$.

Looking at pairs of divisors of 48, we have several possibilities to solve for p_1 and p_2:

$(p_2 + p_1) = 48$

$(p_2 - p_1) = 1$

$p_1 = \frac{47}{2}$

$p_2 = \frac{49}{2}$

$(p_2 + p_1) = 24$

$(p_2 - p_1) = 2$

$p_1 = 11$

$p_2 = 13$

$(p_2 + p_1) = 16$

$(p_2 - p_1) = 3$

$p_1 = \frac{13}{2}$

$p_2 = \frac{19}{2}$

$(p_2 + p_1) = 12$

$(p_2 - p_1) = 4$

$p_1 = 4$

$p_2 = 8$

$(p_2 + p_1) = 8$

$(p_2 - p_1) = 6$

$p_1 = 1$

$p_2 = 7$

The only solution (p_1, p_2) where both numbers are primes is $(11, 13)$.

Therefore the number of positive integers n that satisfy both statements is

See Also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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2005 AMC 10A Problems/Problem 25

Contents

Problem

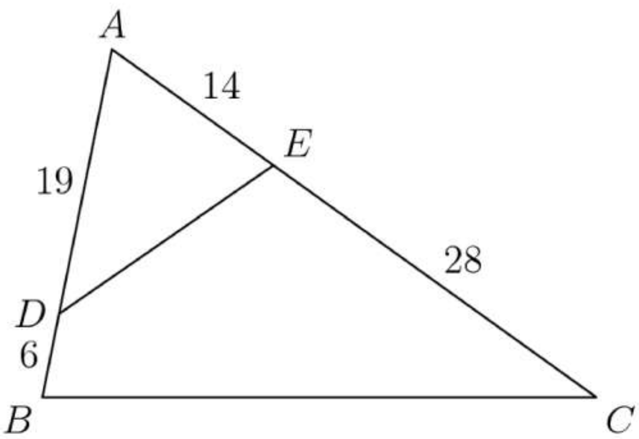
In ABC we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on AB and AC respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$?

$\mathrm{(A)}\ \frac{266}{1521}\qquad\mathrm{(B)}\ \frac{19}{75}\qquad\mathrm{(C)}\ \frac{1}{3}\qquad\mathrm{(D)}\ \frac{19}{56}\qquad\mathrm{(E)}\ 1$

Solution (no trig)

We have that

$$\frac{[ADE]}{[ABC]} = \frac{AD}{AB} \cdot \frac{AE}{AC} = \frac{19}{25} \cdot \frac{14}{42} = \frac{19}{75}.$$



But $[BCED] = [ABC] - [ADE]$, so

$$\begin{aligned}\frac{[ADE]}{[BCED]} &= \frac{[ADE]}{[ABC]-[ADE]}\\ &= \frac{1}{[ABC]/[ADE]-1}\\ &= \frac{1}{75/19-1}\\ &= \boxed{\frac{19}{56}}.\end{aligned}$$

The answer is therefore D.

Solution (trig)

The area of a triangle is $\frac{1}{2}bc\sin A$.

Using this formula:

$$[ADE] = \frac{1}{2} \cdot 19 \cdot 14 \cdot \sin A = 133\sin A$$

$$[ABC] = \frac{1}{2} \cdot 25 \cdot 42 \cdot \sin A = 525\sin A$$

Since the area of $BCED$ is equal to the area of ABC minus the area of ADE,
$[BCED] = 525\sin A - 133\sin A = 392\sin A$.

Therefore, the desired ratio is $\frac{133\sin A}{392\sin A} = \frac{19}{56} \Longrightarrow \mathrm{(D)}$

Note: $BC = 39$ was not used in this problem

See also

2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
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