2007 AMC 12A Problems/Problem 1

The following problem is from both the 2007 AMC 12A #1 and 2007 AMC 10A #1, so both problems redirect to this page.

Problem

One ticket to a show costs $\$20$ at full price. Susan buys 4 tickets using a coupon that gives her a 25% discount. Pam buys 5 tickets using a coupon that gives her a 30% discount. How many more dollars does Pam pay than Susan?

$\mathrm{(A)}\ 2\qquad \mathrm{(B)}\ 5\qquad \mathrm{(C)}\ 10\qquad \mathrm{(D)}\ 15\qquad \mathrm{(E)}\ 20$

Solution 1

P = the amount Pam spent S = the amount Susan spent

- $P = 5 \cdot (20 \cdot .7) = 70$
- $S = 4 \cdot (20 \cdot .75) = 60$

Pam pays 10 more dollars than Susan $\Rightarrow \boxed{\mathrm{C}}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by First question	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by First question	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_12A_Problems/Problem_1&oldid=81140"

Category: Introductory Algebra Problems

2007 AMC 10A Problems/Problem 2

Problem

Define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$. What is $\frac{6@2}{6\#2}$?

$\text{(A)}\ -\frac{1}{2} \qquad \text{(B)}\ -\frac{1}{4} \qquad \text{(C)}\ \frac{1}{8} \qquad \text{(D)}\ \frac{1}{4} \qquad \text{(E)}\ \frac{1}{2}$

Solution

$$\frac{6@2}{6\#2} = \frac{(6)\times(2)-(2)^2}{(6)+(2)-(6)\cdot(2)^2} = \frac{8}{-16} = \frac{-1}{2} \Rightarrow \text{(A)}$$

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2007 AMC 10A Problems/Problem 3

Problem

An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise?

$\mathrm{(A)}\ 0.5\qquad\mathrm{(B)}\ 1\qquad\mathrm{(C)}\ 1.5\qquad\mathrm{(D)}\ 2\qquad\mathrm{(E)}\ 2.5$

Solution

The volume of the brick is $40 \times 20 \times 10 = 8000$. Thus the water volume rose $8000 = 100 \times 40 \times h \Longrightarrow h = 2\ \mathrm{(D)}$.

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2007 AMC 10A Problems/Problem 4

Problem

The larger of two consecutive odd integers is three times the smaller. What is their sum?

$\mathrm{(A)}\ 4\qquad\mathrm{(B)}\ 8\qquad\mathrm{(C)}\ 12\qquad\mathrm{(D)}\ 16\qquad\mathrm{(E)}\ 20$

Solution

Let the two consecutive odd integers be a, $a+2$. Then $a+2 = 3a$, so $a = 1$ and their sum is $4\ \mathrm{(A)}$.

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2007 AMC 10A Problems/Problem 5

Problem

A school store sells 7 pencils and 8 notebooks for $\$4.15$. It also sells 5 pencils and 3 notebooks for $\$1.77$. How much do 16 pencils and 10 notebooks cost?

$\mathrm{(A)}\$1.76 \qquad \mathrm{(B)}\$5.84 \qquad \mathrm{(C)}\$6.00 \qquad \mathrm{(D)}\$6.16 \qquad \mathrm{(E)}\$6.32$

Solution

We let $p =$ cost of pencils in cents, $n =$ number of notebooks in cents. Then

$$\begin{align*} 7p + 8n = 415 &\Longrightarrow 35p + 40n = 2075 \\ 5p + 3n = 177 &\Longrightarrow 35p + 21n = 1239 \end{align*}$$

Subtracting these equations yields $19n = 836 \Longrightarrow n = 44$. Backwards solving gives $p = 9$. Thus the answer is $16p + 10n = 584\ \mathrm{(B)}$.

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Categories: Introductory Algebra Problems | Articles with dollar signs

2007 AMC 10A Problems/Problem 6

Problem

At Euclid High School, the number of students taking the AMC 10 was 60 in 2002, 66 in 2003, 70 in 2004, 76 in 2005, 78 and 2006, and is 85 in 2007. Between what two consecutive years was there the largest percentage increase?

$\mathrm{(A)}\ 2002\ \text{and}\ 2003 \qquad \mathrm{(B)}\ 2003\ \text{and}\ 2004 \qquad \mathrm{(C)}\ 2004\ \text{and}\ 2005 \qquad \mathrm{(D)}\ 2005\ \text{and}\ 2006 \qquad \mathrm{(E)}\ 2006\ \text{and}\ 2007$

Solution

We compute the percentage increases:

1. $\frac{66-60}{60} = 10\%$
2. $\frac{70-66}{66} \approx 6\%$
3. $\frac{76-70}{70} \approx 8.6\%$
4. $\frac{78-76}{76} \approx 2.6\%$
5. $\frac{85-78}{78} \approx 9\%$

The answer is $\mathrm{(A)}$.

In fact, the answer follows directly from examining the differences between each year. The largest differences are $6, 6, 7$, and it is easy to see that due to the decreased starting number of students in 2002 that that will be our answer.

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 5

The following problem is from both the 2007 AMC 12A #5 and 2007 AMC 10A #7, so both problems redirect to this page.

Problem

Last year Mr. Jon Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of $\$10500$ for both taxes. How many dollars was his inheritance?

$\mathrm{(A)}\ 30000 \qquad \mathrm{(B)}\ 32500 \qquad \mathrm{(C)}\ 35000 \qquad \mathrm{(D)}\ 37500 \qquad \mathrm{(E)}\ 40000$

Solution

After paying his taxes, he has $0.8*0.9=0.72$ of the inheritance left. Since 10500 is 0.28 of the inheritance, the whole inheritance is $\frac{10500}{0.28}=37500\ \mathrm{(D)}$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_12A_Problems/Problem_5&oldid=71469"

Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 6

The following problem is from both the 2007 AMC 12A #6 and 2007 AMC 10A #8, so both problems redirect to this page.

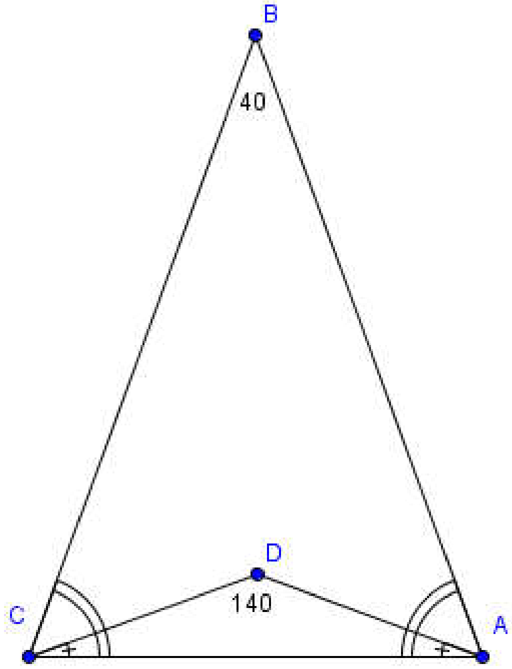
Contents

Problem

Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of angle BAD?

$\mathrm{(A) \ 20 } \qquad \mathrm{(B) \ 30 } \qquad \mathrm{(C) \ 40 } \qquad \mathrm{(D) \ 50 } \qquad \mathrm{(E) \ 60 }$

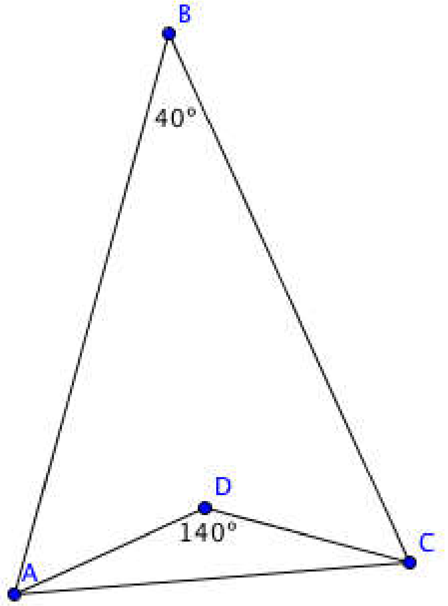
Solution 1



We angle chase, and find out that:

- $DAC = \frac{180 - 140}{2} = 20$
- $BAC = \frac{180 - 40}{2} = 70$
- $BAD = BAC - DAC = 50\ \mathrm{(D)}$

Solution 2



Since triangle ABC is isosceles we know that angle $\angle BAC = \angle BCA$.

Also since triangle ADC is isosceles we know that $\angle DAC = \angle DCA$.

This implies that $\angle BAD = \angle BCD$.

Then the sum of the angles in quadrilateral $ABCD$ is $40 + 220 + 2\angle BAD = 360$.

Solving the equation we get $\angle BAD = 50$.

Therefore the answer is (D).

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Geometry Problems

2007 AMC 10A Problems/Problem 9

Contents

Problem

Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab?

$\mathrm{(A)}\ -60 \qquad \mathrm{(B)}\ -17 \qquad \mathrm{(C)}\ 9 \qquad \mathrm{(D)}\ 12 \qquad \mathrm{(E)}\ 60$

Solution 1

$$81^{b+2} = 3^{4(b+2)} = 3^a \Longrightarrow a = 4b+8$$

And

$$125^b = 5^{3b} = 5^{a-3} \Longrightarrow a-3 = 3b$$

Substitution gives $4b+8-3 = 3b \Longrightarrow b = -5$, and solving for a yields -12. Thus $ab = 60\ \mathrm{(E)}$.

Solution 2

Simplify equation 1 which is 3^a=81^b+2, to 3^a=3^4(b+2), which equals 3^a=3^4b+8.

And

Simplify equation 2 which is 125^b=5^a-3, to 5^3(b)=5^a-3, which equals 5^3b=5^a-3.

Now, eliminate the bases from the simplified equations 1 and 2 to arrive at a=4b+8 and 3b=a-3. Rewrite equation 2 so that it is in terms of a. That would be a=3b+3.

Since both equations are equal to a, and a and b are the same number for both problems, set the equations equal to each other. 4b+8=3b+3 b=-5

Now plug b, which is (-5) back into one of the two earlier equations. 4(-5)+8=a -20+8=a a=-12

(-12)(-5)=60

Therefore the correct answer is E

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 10A Problems/Problem 10

Contents

Problem

The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

$\mathrm{(A)}\ 2 \qquad \mathrm{(B)}\ 3 \qquad \mathrm{(C)}\ 4 \qquad \mathrm{(D)}\ 5 \qquad \mathrm{(E)}\ 6$

Solution 1

Let n be the number of children. Then the total ages of the family is $48+16(n+1)$, and the total number of people in the family is $n+2$. So

$$20 = \frac{48+16(n+1)}{n+2} \Longrightarrow 20n+40 = 16n+64 \Longrightarrow n = 6\ \mathrm{(E)}.$$

Solution 2

Let x be number of children+the mom. The father, who is 48, plus the number of kids and mom divided by the number of kids and mom plus 1 (for the dad)=20. This is because the average age of the entire family is 20. Basically, this looks like 48+16x/x+1=20 48+16x=20x+20 4x=28 x=7

7 people - 1 mom = 6 children.

E is the answer

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_10A_Problems/Problem_10&oldid=79749"

Category: Introductory Algebra Problems

2007 AMC 10A Problems/Problem 11

Contents

Problem

The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

$\text{(A)}\ 14 \qquad \text{(B)}\ 16 \qquad \text{(C)}\ 18 \qquad \text{(D)}\ 20 \qquad \text{(E)}\ 24$

Solution

The sum of the numbers on the top face of a cube is equal to the sum of the numbers on the bottom face of the cube; these 8 numbers represent all of the vertices of the cube. Thus the answer is $\frac{1+2+\cdots+8}{2} = 18\ \mathrm{(C)}$.

Solution 2

Consider a number on a vertex. It will be counted in 3 different faces, the ones it is on. Therefore, each number $1,2,\cdots,7,8$ will be added into the total sum 3 times. Therefore, our total sum is $3(1+2+\cdots+8)=108.$ Finally, since there are 6 faces, our common sum is $\frac{108}{6}=\mathrm{(C)} 18.$

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_10A_Problems/Problem_11&oldid=74134"

Category: Introductory Combinatorics Problems

2007 AMC 10A Problems/Problem 12

Problem

Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

$\mathrm{(A)}\ 56 \qquad \mathrm{(B)}\ 58 \qquad \mathrm{(C)}\ 60 \qquad \mathrm{(D)}\ 62 \qquad \mathrm{(E)}\ 64$

Solution

Each tourist has to pick in between the 2 guides, so for 6 tourists there are 2^6 possible groupings. However, since each guide must take at least one tourist, we subtract the 2 cases where a guide has no tourist. Thus the answer is $2^6 - 2 = 62\ \mathrm{(D)}$.

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_10A_Problems/Problem_12&oldid=54313"

Category: Introductory Combinatorics Problems

2007 AMC 12A Problems/Problem 9

The following problem is from both the 2007 AMC 12A #9 and 2007 AMC 10A #13, so both problems redirect to this page.

Problem

Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or else he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium?

$\mathrm{(A)}\ \frac 23\qquad \mathrm{(B)}\ \frac 34\qquad \mathrm{(C)}\ \frac 45\qquad \mathrm{(D)}\ \frac 56\qquad \mathrm{(E)}\ \frac 78$

Solution

Let the distance from Yan's initial position to the stadium be a and the distance from Yan's initial position to home be b. We are trying to find b/a, and we have the following identity given by the problem:

$$a = b + \frac{a+b}{7}$$
$$\frac{6a}{7} = \frac{8b}{7} \Longrightarrow 6a = 8b$$

Thus $b/a = 6/8 = 3/4$ and the answer is $\mathrm{(B)}\ \frac{3}{4}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 12	Followed by Problem 14
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_12A_Problems/Problem_9&oldid=53714"

Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 10

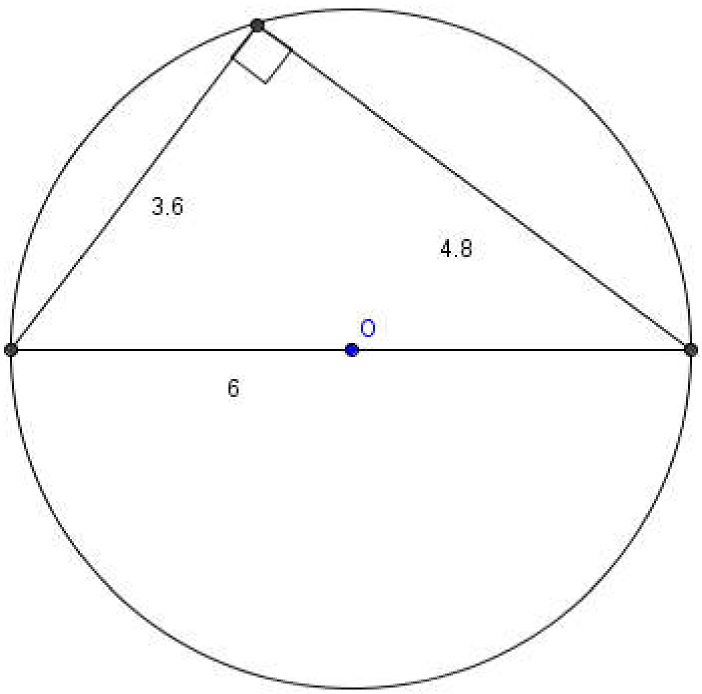
The following problem is from both the 2007 AMC 12A #10 and 2007 AMC 10A #14, so both problems redirect to this page.

Problem

A triangle with side lengths in the ratio $3:4:5$ is inscribed in a circle with radius 3. What is the area of the triangle?

$\mathrm{(A)}\ 8.64\qquad\mathrm{(B)}\ 12\qquad\mathrm{(C)}\ 5\pi\qquad\mathrm{(D)}\ 17.28\qquad\mathrm{(E)}\ 18$

Solution



Since 3-4-5 is a Pythagorean triple, the triangle is a right triangle. Since the hypotenuse is a diameter of the circumcircle, the hypotenuse is $2r = 6$. Then the other legs are $\frac{24}{5} = 4.8$ and $\frac{18}{5} = 3.6$. The area is $\frac{4.8 \cdot 3.6}{2} = 8.64\ \mathrm{(A)}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

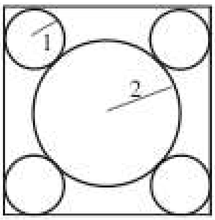
2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 10A Problems/Problem 15

Contents

Problem

Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius 2, as shown. What is the area of the square?



$\text{(A)}\ 32 \qquad \text{(B)}\ 22 + 12\sqrt {2}\qquad \text{(C)}\ 16 + 16\sqrt {3}\qquad \text{(D)}\ 48 \qquad \text{(E)}\ 36 + 16\sqrt {2}$

Solution 1

The diagonal has length $\sqrt{2}+1+2+2+1+\sqrt{2}=6+2\sqrt{2}$. Therefore the sides have length $2+3\sqrt{2}$, and the area is

$$A=(2+3\sqrt{2})^2=4+6\sqrt{2}+6\sqrt{2}+18=22+12\sqrt{2}\Rightarrow \mathrm{(B)}$$

Solution 2

Extend two radii from the larger circle to the centers of the two smaller circles above. This forms a right triangle of sides $3, 3, 3\sqrt{2}$. The length of the hypotenuse of the right triangle plus twice the radius of the smaller circle is equal to the side of the square. It follows, then

$$A=(2+3\sqrt{2})^2=22+12\sqrt{2}\Rightarrow \mathrm{(B)}$$

See Also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 12A Problems/Problem 12

The following problem is from both the 2007 AMC 12A #12 and 2007 AMC 10A #16, so both problems redirect to this page.

Problem

Integers $a, b, c,$ and d, not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

$\mathrm{(A)}\ \frac{3}{8} \qquad \mathrm{(B)}\ \frac{7}{16} \qquad \mathrm{(C)}\ \frac{1}{2} \qquad \mathrm{(D)}\ \frac{9}{16} \qquad \mathrm{(E)}\ \frac{5}{8}$

Solution

The only times when $ad - bc$ is even is when ad and bc are of the same parity. The chance of ad being odd is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, so it has a $\frac{3}{4}$ probability of being even. Therefore, the probability that $ad - bc$ will be even is $\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{5}{8}\ \mathrm{(E)}$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 15	Followed by Problem 17
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Combinatorics Problems

2007 AMC 10A Problems/Problem 17

Problem

Suppose that m and n are positive integers such that $75m = n^3$. What is a minimum possible value of $m + n$?

$\text{(A)}\ 15 \qquad \text{(B)}\ 30 \qquad \text{(C)}\ 50 \qquad \text{(D)}\ 60 \qquad \text{(E)}\ 5700$

Solution

$3 \cdot 5^2 m$ must be a perfect cube, so each power of a prime in the factorization for $3 \cdot 5^2 m$ must be divisible by 3. Thus the minimum value of m is $3^2 \cdot 5 = 45$, which makes $n = \sqrt[3]{3^3 \cdot 5^3} = 15$. These sum to $60\ \mathrm{(D)}$.

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

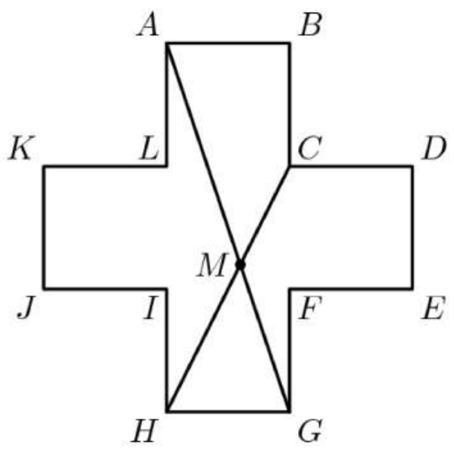
Categories: Introductory Algebra Problems | Introductory Number Theory Problems

2007 AMC 10A Problems/Problem 18

Contents

Problem

Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that $\overline{AG}$ and $\overline{CH}$ meet at M. What is the area of quadrilateral $ABCM$?



$\text{(A)}\ \frac{44}{3}\qquad\text{(B)}\ 16\qquad\text{(C)}\ \frac{88}{5}\qquad\text{(D)}\ 20\qquad\text{(E)}\ \frac{62}{3}$

Solution

Solution 1

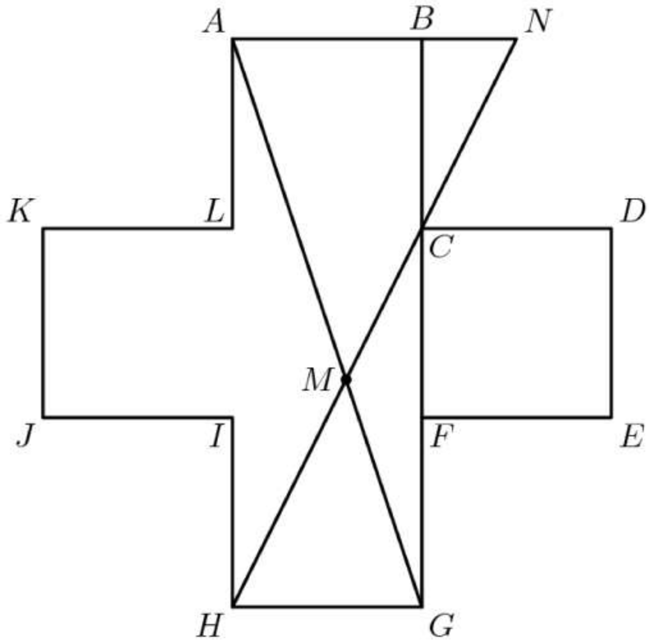
We can obtain the solution by calculating the area of rectangle $ABGH$ minus the combined area of triangles $\triangle AHG$ and $\triangle CGM$.

We know that triangles $\triangle AMH$ and $\triangle GMC$ are similar because $\overline{AH} \parallel \overline{CG}$. Also, since $\frac{AH}{CG} = \frac{3}{2}$, the ratio of the distance from M to $\overline{AH}$ to the distance from M to $\overline{CG}$ is also $\frac{3}{2}$. Solving with the fact that the distance from $\overline{AH}$ to $\overline{CG}$ is 4, we see that the distance from M to $\overline{CG}$ is $\frac{8}{5}$.

The area of $\triangle AHG$ is simply $\frac{1}{2}\cdot 4\cdot 12 = 24$, the area of $\triangle CGM$ is $\frac{1}{2}\cdot\frac{8}{5}\cdot 8 = \frac{32}{5}$, and the area of rectangle $ABGH$ is $4\cdot 12 = 48$.

Taking the area of rectangle $ABGH$ and subtracting the combined area of $\triangle AHG$ and $\triangle CGM$ yields $48-(24+\frac{32}{5}) = \boxed{\frac{88}{5}}\ \text{(C)}$.

Solution 2



Extend AB and CH and call their intersection N.

The triangles CBN and CGH are clearly similar with ratio $1:2$, hence $BN=2$ and thus $AN=6$. The area of the triangle BCN is $\frac{2\cdot 4}{2}=4$.

The triangles MAN and MGH are similar as well, and we now know that the ratio of their dimensions is $AN:GH=6:4=3:2$.

Draw altitudes from M onto AN and GH, let their feet be M_1 and M_2. We get that $MM_1:MM_2=3:2$. Hence $MM_1=\frac{3}{5}\cdot 12=\frac{36}{5}$. (An alternate way is by seeing that the set-up AHGCM is similar to the 2 pole problem(http://www.artofproblemsolving.com/wiki/index.php/1951_AHSME_Problems/Problem_30). Therefore, MM_2 must be $\frac{1}{\frac{1}{8}+\frac{1}{12}}=\frac{24}{5}$, by the harmonic mean. Thus, MM_1 must be $\frac{36}{5}$.)

Then the area of AMN is $\frac{1}{2}\cdot AN\cdot MM_1=\frac{108}{5}$, and the area of $ABCM$ can be obtained by subtracting the area of BCN, which is 4. Hence the answer is $\frac{108}{5}-4=\boxed{\frac{88}{5}}$.

Solution 3

We can use coordinates to solve this. Let $H=(0,0)$. Thus, we have $A=(0,12), C=(4,8), G=(4,0)$. Therefore, AG has equation $-3x+12=y$ and HC has equation $2x=y$. Solving, we have $M=(12/5,24/5)$. Using Shoelace Theorem (or you could connect LC and solve for the resulting triangle + trapezoid areas), we find $[ABCM]=\boxed{\text{(C) }\frac{88}{5}}$.

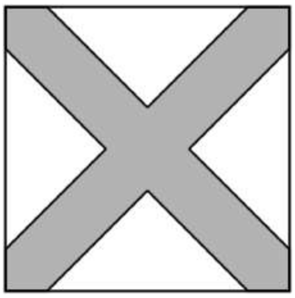
See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 10A Problems/Problem 19

Problem

A paint brush is swept along both diagonals of a square to produce the symmetric painted area, as shown. Half the area of the square is painted. What is the ratio of the side length of the square to the brush width?



$\mathrm{(A)}\ 2\sqrt{2}+1 \qquad \mathrm{(B)}\ 3\sqrt{2} \qquad \mathrm{(C)}\ 2\sqrt{2}+2 \qquad \mathrm{(D)}\ 3\sqrt{2}+1 \qquad \mathrm{(E)}\ 3\sqrt{2}+2$

Solution

Solution 1

Without loss of generality, let the side length of the square be 1 unit. The area of the painted area is $\frac{1}{2}$ of the area of the larger square, so the total unpainted area is also $\frac{1}{2}$. Each of the 4 unpainted triangle has area $\frac{1}{8}$. It is easy to tell that these triangles are isosceles right triangles, so let a be the side length of one of the smaller triangles:

$$\frac{a^2}{2}=\frac{1}{8}\rightarrow a=\frac{1}{2}$$

The diagonal of the triangle is $\frac{\sqrt{2}}{2}$. The corners of the painted areas are also isosceles right triangles with side length $\frac{1-\frac{\sqrt{2}}{2}}{2}=\frac{1}{2}-\frac{\sqrt{2}}{4}$. Its diagonal is equal to the width of the paint, and is $\frac{\sqrt{2}}{2}-\frac{1}{2}$. The answer we are looking for is thus $\frac{1}{\frac{\sqrt{2}}{2}-\frac{1}{2}}$. Multiply the numerator and the denominator by $\frac{\sqrt{2}}{2}+\frac{1}{2}$ to simplify, and you get $\frac{\frac{\sqrt{2}}{2}+\frac{1}{2}}{\frac{2}{4}-\frac{1}{4}}$ or $4\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right)$ which is $\boxed{2\sqrt{2}+2}\rightarrow C$.

Solution 2

Again, have the length of the square equal to 1 and let the width of each individual stripe be n. Note that you can split each stripe into two rectangles and two isosceles right triangles at the corners. Then the area of each stripe is $n(\sqrt{2}-\frac{n}{2})=\sqrt{2}n-\frac{n^2}{2}$. The area covered by the two total stripes is twice the area of one stripe, minus the area in the intersection of the stripes, which is a square with side length n. This area is equal to $\frac{1}{2}$ So:

$$2\sqrt{2}n-2n^2=\frac{1}{2}\rightarrow -2n^2+2\sqrt{2}n-\frac{1}{2}=0.$$

By the quadratic formula,

$$n=\frac{-2\sqrt{2}\pm\sqrt{(2\sqrt{2})^2-4(-2)(-\frac{1}{2})}}{2(-2)}\rightarrow n=\frac{-2\sqrt{2}\pm 2}{-4}\rightarrow n=\frac{\sqrt{2}\pm 1}{2}$$

It's easy to tell that $\frac{\sqrt{2}+1}{2}$ is too large, so $n=\frac{\sqrt{2}-1}{2}$. We want to find $\frac{1}{n}$, and $\frac{1}{n}=\frac{2}{\sqrt{2}-1}$. Multiply the numerator and the denominator by $\sqrt{2}+1$,

$$\frac{1}{n}=\frac{2(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}=\boxed{2\sqrt{2}+2}\rightarrow C$$

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Geometry Problems

2007 AMC 10A Problems/Problem 20

Problem

Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

$\mathrm{(A)}\ 164 \qquad \mathrm{(B)}\ 172 \qquad \mathrm{(C)}\ 192 \qquad \mathrm{(D)}\ 194 \qquad \mathrm{(E)}\ 212$

Solution

Solution 1

Notice that $(a^k + a^{-k})^2 = a^{2k} + a^{-2k} + 2$. Thus
$a^4 + a^{-4} = (a^2 + a^{-2})^2 - 2 = [(a + a^{-1})^2 - 2]^2 - 2 = 194\ \mathrm{(D)}$.

Solution 2

$4a = a^2 + 1$. We apply the quadratic formula to get $a = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$.

Thus $a^4 + a^{-4} = (2+\sqrt{3})^4 + \frac{1}{(2+\sqrt{3})^4} = (2+\sqrt{3})^4 + (2-\sqrt{3})^4$ (so it doesn't matter which root of a we use). Using the binomial theorem we can expand this out and collect terms to get 194.

Solution 3

We know that $a + \frac{1}{a} = 4$. We can square both sides to get $a^2 + \frac{1}{a^2} + 2 = 16$, so $a^2 + \frac{1}{a^2} = 14$. Squaring both sides again gives $a^4 + \frac{1}{a^4} + 2 = 14^2 = 196$, so $a^4 + \frac{1}{a^4} = \boxed{194}$.

Solution 4

We let a and $1/a$ be roots of a certain quadratic. Specifically $x^2 - 4x + 1 = 0$. We use Newton's Sums given the coefficients to find S_4. $S_4 = \boxed{194}$

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_10A_Problems/Problem_20&oldid=74979"

Category: Introductory Algebra Problems

2007 AMC 10A Problems/Problem 21

Problem

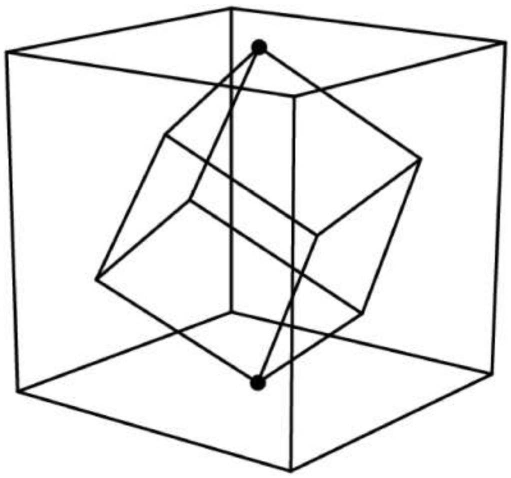
A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

$\text{(A)}\ 3 \qquad \text{(B)}\ 6 \qquad \text{(C)}\ 8 \qquad \text{(D)}\ 9 \qquad \text{(E)}\ 12$

Contents

Solution

Solution 1



We rotate the smaller cube around the sphere such that two opposite vertices of the cube are on opposite faces of the larger cube. Thus the main diagonal of the smaller cube is the side length of the outer square.

Let S be the surface area of the inner square. The ratio of the areas of two similar figures is equal to the square of the ratio of their sides. As the diagonal of a cube has length $s\sqrt{3}$ where s is a side of the cube, the ratio of a side of the inner square to that of the outer square (and the side of the outer square = the diagonal of the inner square), we have $\frac{S}{24} = \left(\frac{1}{\sqrt{3}}\right)^2$. Thus $S = 8 \Rightarrow \mathrm{(C)}$.

Solution 2 (computation)

The area of each face of the outer cube is $\frac{24}{6} = 4$, and the edge length of the outer cube is 2. This is also the diameter of the sphere, and thus the length of a long diagonal of the inner cube.

A long diagonal of a cube is the hypotenuse of a right triangle with a side of the cube and a face diagonal of the cube as legs. If a side of the cube is x, we see that $2 = \sqrt{x^2+(\sqrt{2}x)^2} \Rightarrow x = \frac{2}{\sqrt{3}}$.

Thus the surface area of the inner cube is $6x^2 = 6\left(\frac{2}{\sqrt{3}}\right)^2 = 8$.

2007 AMC 12A Problems/Problem 11

The following problem is from both the 2007 AMC 12A #11 and 2007 AMC 10A #22, so both problems redirect to this page.

Problem

A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with the terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime factor that always divides S?

$\mathrm{(A)}\ 3\qquad \mathrm{(B)}\ 7\qquad \mathrm{(C)}\ 13\qquad \mathrm{(D)}\ 37\qquad \mathrm{(E)}\ 43$

Solution

A given digit appears as the hundreds digit, the tens digit, and the units digit of a term the same number of times. Let k be the sum of the units digits in all the terms. Then $S=111k=3*37k$, so S must be divisible by $37\ \mathrm{(D)}$. To see that it need not be divisible by any larger prime, the sequence $123, 231, 312$ gives $S=666=2\cdot 3^2\cdot 37$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 21	Followed by Problem 23
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Categories: Introductory Algebra Problems | Introductory Number Theory Problems

2007 AMC 10A Problems/Problem 23

Problem

How many ordered pairs (m, n) of positive integers, with $m \geq n$, have the property that their squares differ by 96?

$\mathrm{(A)}\ 3 \qquad \mathrm{(B)}\ 4 \qquad \mathrm{(C)}\ 6 \qquad \mathrm{(D)}\ 9 \qquad \mathrm{(E)}\ 12$

Solution

$$m^2 - n^2 = (m+n)(m-n) = 96 = 2^5 \cdot 3$$

For every two factors $xy = 96$, we have $m+n = x, m-n = y \Longrightarrow m = \frac{x+y}{2}, n = \frac{x-y}{2}$. Since $m \geq n > 0$, $x+y \geq x-y > 0$, from which it follows that the number of ordered pairs (m,n) is given by the number of ordered pairs $(x,y) : xy = 96, x > y > 0$. There are $(5+1)(1+1) = 12$ factors of 96, which give us six pairs (x,y). However, since m,n are positive integers, we also need that $\frac{x+y}{2}, \frac{x-y}{2}$ are positive integers, so x and y must have the same parity. Thus we exclude the factors $(x,y) = (1,96)(3,32)$, and we are left with four pairs $\mathrm{(B)}$.

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

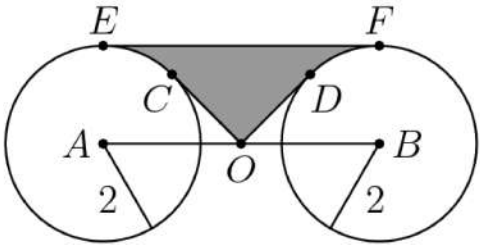
Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_10A_Problems/Problem_23&oldid=54320"

Category: Introductory Number Theory Problems

2007 AMC 10A Problems/Problem 24

Problem

Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of $\overline{AB}$, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and EF is a common tangent. What is the area of the shaded region $ECODF$?



$\text{(A)}\ \frac{8\sqrt{2}}{3}\qquad\text{(B)}\ 8\sqrt{2}-4-\pi\qquad\text{(C)}\ 4\sqrt{2}\qquad\text{(D)}\ 4\sqrt{2}+\frac{\pi}{8}\qquad\text{(E)}\ 8\sqrt{2}-2-\frac{\pi}{2}$

Solution

The area we are trying to find is simply $ABFE - (\widehat{AEC} + \triangle ACO + \triangle BDO + \widehat{BFD})$. Obviously, $\overline{EF} \parallel \overline{AB}$. Thus, $ABFE$ is a rectangle, and so its area is $b \times h = 2 \times (AO + OB) = 2 \times 2(2\sqrt{2}) = 8\sqrt{2}$.

Since $\overline{OC}$ is tangent to circle A, $\triangle ACO$ is a right triangle. We know $AO = 2\sqrt{2}$ and $AC = 2$, so $\triangle ACO$ is isosceles, a 45-45 right triangle, and has $\overline{CO}$ with length 2. The area of $\triangle ACO = \frac{1}{2}bh = 2$. By symmetry, $\triangle ACO \cong \triangle BDO$, and so the area of $\triangle BDO$ is also 2.

$\widehat{AEC}$ (or $\widehat{BFD}$, for that matter) is $\frac{1}{8}$ the area of its circle. Thus $\widehat{AEC}$ and $\widehat{BFD}$ both have an area of $\frac{\pi}{2}$.

Plugging all of these areas back into the original equation yields
$$8\sqrt{2} - (\frac{\pi}{2} + 2 + 2 + \frac{\pi}{2}) = 8\sqrt{2} - (4 + \pi) = \boxed{8\sqrt{2} - 4 - \pi}\ \text{(B)}.$$

See also

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 23	Followed by Problem 25
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 12A Problems/Problem 22

The following problem is from both the 2007 AMC 12A #22 and 2007 AMC 10A #25, so both problems redirect to this page.

Problem

For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007?$

$\mathrm{(A)}\ 1 \qquad \mathrm{(B)}\ 2 \qquad \mathrm{(C)}\ 3 \qquad \mathrm{(D)}\ 4 \qquad \mathrm{(E)}\ 5$

Contents

Solution

Solution 1

For the sake of notation let $T(n) = n + S(n) + S(S(n))$. Obviously $n<2007$. Then the maximum value of $S(n) + S(S(n))$ is when $n = 1999$, and the sum becomes $28 + 10 = 38$. So the minimum bound is 1969. We do casework upon the tens digit:

Case 1: $196u \Longrightarrow u = 9$. Easy to directly disprove.

Case 2: $197u$. $S(n) = 1 + 9 + 7 + u = 17 + u$, and $S(S(n)) = 8 + u$ if $u \le 2$ and $S(S(n)) = 2 + (u-3) = u - 1$ otherwise.

Subcase a: $T(n) = 1970 + u + 17 + u + 8 + u = 1995 + 3u = 2007 \Longrightarrow u = 4$. This exceeds our bounds, so no solution here.
Subcase b: $T(n) = 1970 + u + 17 + u + u - 1 = 1986 + 3u = 2007 \Longrightarrow u = 7$. First solution.

Case 3: $198u$. $S(n) = 18 + u$, and $S(S(n)) = 9 + u$ if $u \le 1$ and $2 + (u-2) = u$ otherwise.

Subcase a: $T(n) = 1980 + u + 18 + u + 9 + u = 2007 + 3u = 2007 \Longrightarrow u = 0$. Second solution.
Subcase b: $T(n) = 1980 + u + 18 + u + u = 1998 + 3u = 2007 \Longrightarrow u = 3$. Third solution.

Case 4: $199u$. But $S(n) > 19$, and $n + S(n)$ clearly sum to > 2007.

Case 5: $200u$. So $S(n) = 2 + u$ and $S(S(n)) = 2 + u$ (recall that $n < 2007$), and $2000 + u + 2 + u + 2 + u = 2004 + 3u = 2007 \Longrightarrow u = 1$. Fourth solution.

In total we have $4 \mathrm{(D)}$ solutions, which are $1977, 1980, 1983,$ and 2001.

Solution 2

Clearly, $n>1950$. We can break this into three cases:

Case 1: $n\geq 2000$

Inspection gives $n=2001$.

Case 2: $n<2000$, $n=19xy$(not to be confused with $19*x*y$), $x+y<10$

If you set up an equation, it reduces to

$$4x + y = 32$$

which has as its only solution satisfying the constraints $x = 8,\ y = 0$.

Case 3: $n < 2000,\ n = 19xy,\ x + y \geq 10$

This reduces to

$4x + y = 35$. The only two solutions satisfying the constraints for this equation are $x = 7,\ y = 7$ and $x = 8,\ y = 3$.

The solutions are thus $1977, 1980, 1983, 2001$ and the answer is $(\text{D})\ 4$.

Solution 3

As in Solution 1, we note that $S(n) \leq 28$ and $S(S(n)) \leq 10$.
Obviously, $n \equiv S(n) \equiv S(S(n)) \pmod 9$.
As $2007 \equiv 0 \pmod 9$, this means that $n \bmod 9 \in \{0, 3, 6\}$, or equivalently that $n \equiv S(n) \equiv S(S(n)) \equiv 0 \pmod 3$.

Thus $S(S(n)) \in \{3, 6, 9\}$. For each possible $S(S(n))$ we get three possible $S(n)$.
(E. g., if $S(S(n)) = 6$, then $S(n) = x$ is a number such that $x \leq 28$ and $S(x) = 6$, therefore $S(n) \in \{6, 15, 24\}$.)

For each of these nine possibilities we compute $n_?$ as $2007 - S(n) - S(S(n))$ and check whether $S(n_?) = S(n)$.
We'll find out that out of the 9 cases, in 4 the value $n_?$ has the correct sum of digits.
This happens for $n_? \in \{1977, 1980, 1983, 2001\}$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 21	Followed by Problem 23
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 24	Followed by Last question
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2007_AMC_12A_Problems/Problem_22&oldid=58235"

Category: Introductory Number Theory Problems