2010 AMC 10B Problems/Problem 1

Problem

What is $100(100-3)-(100\cdot100-3)$?

$\textbf{(A)}\ -20,000 \qquad \textbf{(B)}\ -10,000 \qquad \textbf{(C)}\ -297 \qquad \textbf{(D)}\ -6 \qquad \textbf{(E)}\ 0$

Solution

$100(100-3)-(100\cdot100-3) = 10000-300-10000+3 = -300+3 = \boxed{\textbf{(C)}\ -297}$.

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 2

Problem

Makarla attended two meetings during her 9-hour work day. The first meeting took 45 minutes and the second meeting took twice as long. What percent of her work day was spent attending meetings?

$\mathrm{(A) \ } 15 \qquad \mathrm{(B) \ } 20 \qquad \mathrm{(C) \ } 25 \qquad \mathrm{(D) \ } 30 \qquad \mathrm{(E) \ } 35$

Solution

The percentage of her time spent in meetings is the total amount of time spent in meetings divided by the length of her workday.

The total time spent in meetings is $45 \text{ min} + 2*45 \text{ min} = 2 \text{ hours } 15 \text{ min} = 9/4 \text{ hours}$

Therefore, the percentage is

$$\frac{9/4 \text{ hours}}{9 \text{ hours}} = \frac{1}{4} = 25\% = \boxed{\mathrm{(C)} 25}$$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 3

Problem

A drawer contains red, green, blue, and white socks with at least 2 of each color. What is the minimum number of socks that must be pulled from the drawer to guarantee a matching pair?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 9$

Solution

After you draw 4 socks, you can have one of each color, so (according to the pigeonhole principle), if you pull $\boxed{\textbf{(C)}\ 5}$ then you will be guaranteed a matching pair.

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 4

Problem

For a real number x, define $\heartsuit(x)$ to be the average of x and x^2. What is $\heartsuit(1)+\heartsuit(2)+\heartsuit(3)$?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 20$

Solution

The average of two numbers, a and b, is defined as $\frac{a+b}{2}$. Thus the average of x and x^2 would be $\frac{x(x+1)}{2}$. With that said, we need to find the sum when we plug, 1, 2 and 3 into that equation. So:

$$\frac{1(1+1)}{2}+\frac{2(2+1)}{2}+\frac{3(3+1)}{2} = \frac{2}{2}+\frac{6}{2}+\frac{12}{2} = 1+3+6 = \boxed{\textbf{(C)} 10}$$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 5

Problem

A month with 31 days has the same number of Mondays and Wednesdays. How many of the seven days of the week could be the first day of this month?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

Solution

In this month there are four weeks and three remaining days. As long as the last three days and the first four days have the same number of Mondays and Wednesdays, then it works. The number of days the month can start on is $\boxed{\textbf{(B)}\ 3}$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 6

Problem

A circle is centered at O, $\overline{AB}$ is a diameter and C is a point on the circle with $\angle COB = 50^\circ$. What is the degree measure of $\angle CAB$?

$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ 25 \qquad \textbf{(C)}\ 45 \qquad \textbf{(D)}\ 50 \qquad \textbf{(E)}\ 65$

Solution

Assuming we do not already know an inscribed angle is always half of its central angle, we will try a different approach. Since O is the center, OC and OA are radii and they are congruent. Thus, $\triangle COA$ is an isosceles triangle. Also, note that $\angle COB$ and $\angle COA$ are supplementary, then $\angle COA = 180 - 50 = 130^\circ$. Since $\triangle COA$ is isosceles, then $\angle OCA \cong \angle OAC$. They also sum to 50°, so each angle is $\boxed{\textbf{(B)}\ 25}$.

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 7

Contents

Problem

A triangle has side lengths 10, 10, and 12. A rectangle has width 4 and area equal to the area of the triangle. What is the perimeter of this rectangle?

$\textbf{(A)}\ 16 \qquad \textbf{(B)}\ 24 \qquad \textbf{(C)}\ 28 \qquad \textbf{(D)}\ 32 \qquad \textbf{(E)}\ 36$

Solution 1

The triangle is isosceles. The height of the triangle is therefore given by $h = \sqrt{10^2 - (\frac{12}{2})^2} = \sqrt{64} = 8$

Now, the area of the triangle is $\frac{bh}{2} = \frac{12*8}{2} = \frac{96}{2} = 48$

We have that the area of the rectangle is the same as the area of the triangle, namely 48. We also have the width of the rectangle: 4.

The length of the rectangle therefore is: $l = \frac{48}{4} = 12$

The perimeter of the rectangle then becomes: $2l + 2w = 2*12 + 2*4 = 32$

The answer is:

$\boxed{\textbf{(D)}\ 32}$

Solution 2

An alternative way to find the area of the triangle is by using Heron's formula, $A = \sqrt{(s)(s-a)(s-b)(s-c)}$ where s is the semi-perimeter of the triangle (meaning half the perimeter). Here, the semi-perimeter is $(10+10+12)/2 = 16$. Thus the area equals $\sqrt{(16)(16-10)(16-10)(16-12)} = \sqrt{16*6*6*4} = 48$. We know that the width of the rectangle is 4, so $48/4 = 12$, which is the length. The perimeter of the rectangle is $2(4+12) = \boxed{\textbf{(D)}\ 32}$.

(Solution by Flamedragon)

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 8

Problem

A ticket to a school play cost x dollars, where x is a whole number. A group of 9th graders buys tickets costing a total of $\$48$, and a group of 10th graders buys tickets costing a total of $\$64$. How many values for x are possible?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

We see how many common integer factors 48 and 64 share. Of the factors of 48 - $1, 2, 3, 4, 6, 8, 12, 16, 24, 48$; only $1, 2, 4, 8,$ and 16 are factors of 64. So there are $\boxed{\textbf{(E)}\ 5}$ possibilities for the ticket price.

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 9

Contents

Problem

Lucky Larry's teacher asked him to substitute numbers for a, b, c, d, and e in the expression $a-(b-(c-(d+e)))$ and evaluate the result. Larry ignored the parenthese but added and subtracted correctly and obtained the correct result by coincidence. The number Larry sustitued for a, b, c, and d were 1, 2, 3, and 4, respectively. What number did Larry substitude for e?

$\textbf{(A)}\ -5 \qquad \textbf{(B)}\ -3 \qquad \textbf{(C)}\ 0 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 5$

Solution 1

Simplify the expression $a-(b-(c-(d+e)))$. I recommend to start with the innermost parenthesis and work your way out.

So you get:
$a-(b-(c-(d+e)))=a-(b-(c-d-e))=a-(b-c+d+e))=a-b+c-d-e$

Henry substituted a, b, c, d with $1, 2, 3, 4$ respectively.

We have to find the value of e, such that $a-b+c-d-e=a-b-c-d+e$ (the same expression without parenthesis).

Substituting and simplifying we get: $-2-e=-8+e \Rightarrow -2e=-6 \Rightarrow e=3$

So Henry must have used the value 3 for e.

Our answer is $3 \Rightarrow \boxed{\textbf{(D)}}$

Solution 2

Lucky Larry had not been aware of the parenthesis and would have done the following operations:
$1-2-3-4+e=e-8$

The correct way he should have done the operations is:

$$1-(2-(3-(4+e)) = 1-(2-(3-4-e) = 1-(2-(-1-e) = 1-(3+e) = 1-3-e = -e-2$$

Therefore we have the equation $e-8=-e-2 \implies 2e=6 \implies e=3 \Rightarrow \boxed{\textbf{(D)}}$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 10

Problem

Shelby drives her scooter at a speed of 30 miles per hour if it is not raining, and 20 miles per hour if it is raining. Today she drove in the sun in the morning and in the rain in the evening, for a total of 16 miles in 40 minutes. How many minutes did she drive in the rain?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 21 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 27 \qquad \textbf{(E)}\ 30$

Solution

We know that $d = vt$

Since we know that she drove both when it was raining and when it was not and that her total distance traveled is 16 miles.

We also know that she drove a total of 40 minutes which is $\frac{2}{3}$ of an hour.

We get the following system of equations, where x is the time traveled when it was not raining and y is the time traveled when it was raining:

$$\left\{\begin{array}{ccc} 30x + 20y & = & 16 \\ x + y & = & \frac{2}{3} \end{array}\right.$$

Solving the above equations by multiplying the second equation by 30 and subtracting the second equation from the first we get:

$$-10y = -4 \Leftrightarrow y = \frac{2}{5}$$

We know now that the time traveled in rain was $\frac{2}{5}$ of an hour, which is $\frac{2}{5} * 60 = 24$ minutes

So, our answer is $\boxed{\textbf{(C)}\ 24}$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 11

Problem

A shopper plans to purchase an item that has a listed price greater than $\$100$ and can use any one of the three coupons. Coupon A gives 15% off the listed price, Coupon B gives $\$30$ off the listed price, and Coupon C gives 25% off the amount by which the listed price exceeds $\$100$.
Let x and y be the smallest and largest prices, respectively, for which Coupon A saves at least as many dollars as Coupon B or C. What is $y - x$?

$\textbf{(A)}\ 50 \qquad \textbf{(B)}\ 60 \qquad \textbf{(C)}\ 75 \qquad \textbf{(D)}\ 80 \qquad \textbf{(E)}\ 100$

Solution

Let the listed price be $(100 + p)$, where $p > 0$

Coupon A saves us: $0.15(100+p) = (0.15p + 15)$

Coupon B saves us: 30

Coupon C saves us: $0.25p$

Now, the condition is that A has to be greater than or equal to either B or C which give us the following inequalities:

$A \geq B \Rightarrow 0.15p + 15 \geq 30 \Rightarrow p \geq 100$

$A \geq C \Rightarrow 0.15p + 15 \geq 0.25p \Rightarrow p \leq 150$

We see here that the greatest possible value for p is 150, thus $y = 100 + 150 = 250$ and the smallest value for p is 100 so $x = 100 + 100 = 200$.

The difference between y and x is $y - x = 250 - 200 = \boxed{\textbf{(A)}\ 50}$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 12

Problem

At the beginning of the school year, 50% of all students in Mr. Wells' math class answered "Yes" to the question "Do you love math", and 50% answered "No." At the end of the school year, 70% answered "Yes" and 30% answered "No." Altogether, $x\%$ of the students gave a different answer at the beginning and end of the school year. What is the difference between the maximum and the minimum possible values of x?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 20 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 60 \qquad \textbf{(E)}\ 80$

Solution

The minimum possible value occurs when 20% of the students who originally answered "No." answer "Yes." In this case, $x = 20$

The maximum possible value occurs when 60% of the students (which accounts for 30% of the overall student population) who originally answered "Yes." answer "No." and the 100% of the students (which accounts for 50% of the overall student population) who originally answered "No." answer "Yes." In this case, $x = 50 + 30 = 80$

Subtract $80 - 20$ to obtain an answer of $\boxed{\textbf{(D)}\ 60}$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 13

Problem

What is the sum of all the solutions of $x = |2x - |60 - 2x||$?

$\textbf{(A)}\ 32 \qquad \textbf{(B)}\ 60 \qquad \textbf{(C)}\ 92 \qquad \textbf{(D)}\ 120 \qquad \textbf{(E)}\ 124$

Solution

We evaluate this in cases:

Case 1 $x < 30$

When $x < 30$ we are going to have $60 - 2x > 0$. When $x > 0$ we are going to have $|x| > 0 \implies x > 0$ and when $-x > 0$ we are going to have $|x| > 0 \implies -x > 0$. Therefore we have $x = |2x - (60 - 2x)|$. $x = |2x - 60 + 2x| \implies x = |4x - 60|$

Subcase 1 $30 > x > 15$

When $30 > x > 15$ we are going to have $4x - 60 > 0$. When this happens, we can express $|4x - 60|$ as $4x - 60$. Therefore we get $x = 4x - 60 \implies -3x = -60 \implies x = 20$. We check if $x = 20$ is in the domain of the numbers that we put into this subcase, and it is, since $30 > 20 > 15$. Therefore 20 is one possible solution.

Subcase 2 $x < 15$

When $x < 15$ we are going to have $4x - 60 < 0$, therefore $|4x - 60|$ can be expressed in the form $60 - 4x$. We have the equation $x = 60 - 4x \implies 5x = 60 \implies x = 12$. Since 12 is less than 15, 12 is another possible solution. $x = |2x - |60 - 2x||$

Case 2 : $x > 30$

When $x > 30$, $60 - 2x < 0$. When $x < 0$ we can express this in the form $-x$. Therefore we have $-(60 - 2x) = 2x - 60$. This makes sure that this is positive, since we just took the negative of a negative to get a positive. Therefore we have $x = |2x - (2x - 60)|$

$$x = |2x - 2x + 60|$$

$$x = |60|$$

$$x = 60$$

We have now evaluated all the cases, and found the solution to be $\{60, 12, 20\}$ which have a sum of $\boxed{\textbf{(C)}\ 92}$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 14

Problem

The average of the numbers $1, 2, 3, \cdots, 98, 99,$ and x is $100x$. What is x?

$\textbf{(A)}\ \frac{49}{101} \qquad \textbf{(B)}\ \frac{50}{101} \qquad \textbf{(C)}\ \frac{1}{2} \qquad \textbf{(D)}\ \frac{51}{101} \qquad \textbf{(E)}\ \frac{50}{99}$

Solution

We must find the average of the numbers from 1 to 99 and x in terms of x. The sum of all these terms is $\frac{99(100)}{2}+x=99(50)+x$. We must divide this by the total number of terms, which is 100. We get: $\frac{99(50)+x}{100}$. This is equal to $100x$, as stated in the problem. We have: $\frac{99(50)+x}{100}=100x$. We can now cross multiply. This gives

$$100(100x)=99(50)+x \rightarrow 10000x=99(50)+x \rightarrow 9999x=99(50) \rightarrow 101x=50 \rightarrow x=\boxed{\textrm{(B)}\frac{50}{101}}$$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 15

Problem

On a 50-question multiple choice math contest, students receive 4 points for a correct answer, 0 points for an answer left blank, and -1 point for an incorrect answer. Jesse's total score on the contest was 99. What is the maximum number of questions that Jesse could have answered correctly?

$\mathrm{(A)}\ 25 \qquad \mathrm{(B)}\ 27 \qquad \mathrm{(C)}\ 29 \qquad \mathrm{(D)}\ 31 \qquad \mathrm{(E)}\ 33$

Solution

Let a be the amount of questions Jesse answered correctly, b be the amount of questions Jesse left blank, and c be the amount of questions Jesse answered incorrectly. Since there were 50 questions on the contest, $a+b+c=50$. Since his total score was 99, $4a-c=99$. Also, $a+c \le 50 \Rightarrow c \le 50-a$. We can substitute this inequality into the previous equation to obtain another inequality:

$4a-(50-a) \le 99 \Rightarrow 5a \le 149 \Rightarrow a \le \frac{149}{5} = 29.8$. Since a is an integer, the maximum value for a is $\boxed{\mathrm{(C)}\ 29}$.

See Also

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2010 AMC 10B Problems/Problem 16

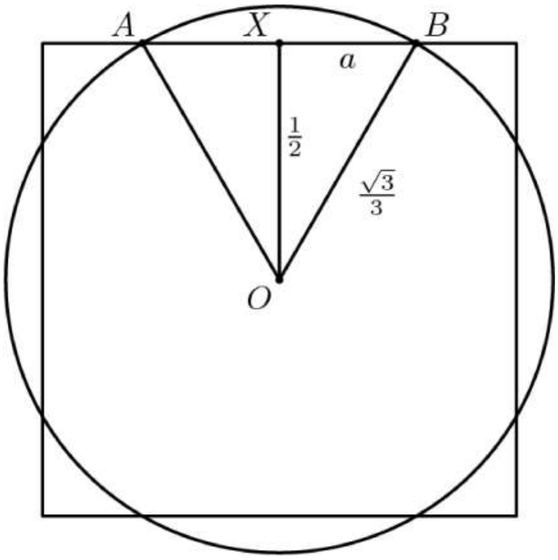
Problem

A square of side length 1 and a circle of radius $\frac{\sqrt{3}}{3}$ share the same center. What is the area inside the circle, but outside the square?

$\textbf{(A)}\ \frac{\pi}{3}-1 \qquad \textbf{(B)}\ \frac{2\pi}{9}-\frac{\sqrt{3}}{3} \qquad \textbf{(C)}\ \frac{\pi}{18} \qquad \textbf{(D)}\ \frac{1}{4} \qquad \textbf{(E)}\ \frac{2\pi}{9}$

Solution

The radius of circle is $\frac{\sqrt{3}}{3}=\sqrt{\frac{1}{3}}$. Half the diagonal of the square is $\frac{\sqrt{1^2+1^2}}{2}=\frac{\sqrt{2}}{2}=\sqrt{\frac12}$. We can see that the circle passes outside the square, but the square is NOT completely contained in the circle Therefore the picture will look something like this:



Then we proceed to find: 4 * (area of sector marked off by the two radii - area of the triangle with sides on the square and the two radii).

First we realize that the radius perpendicular to the side of the square between the two radii marking off the sector splits AB in half. Let this half-length be a. Also note that $OX=\frac{1}{2}$ because it is half the sidelength of the square. Because this is a right triangle, we can use the Pythagorean Theorem to solve for a.

$$a^2+\left(\frac{1}{2}\right)^2=\left(\frac{\sqrt{3}}{3}\right)^2$$

Solving, $a=\frac{\sqrt{3}}{6}$ and $2a=\frac{\sqrt{3}}{3}$. Since $AB=AO=BO$, $\triangle AOB$ is an equilateral triangle and the central angle is 60°. Therefore the sector has an area $\pi\left(\frac{\sqrt{3}}{3}\right)^2\left(\frac{60}{360}\right)=\frac{\pi}{18}$.

Now we turn to the triangle. Since it is equilateral, we can use the formula for the area of an equilateral triangle which is

$$\frac{s^2\sqrt{3}}{4} = \frac{\frac{1}{3}\sqrt{3}}{4} = \frac{\sqrt{3}}{12}$$

Putting it together, we get the answer to be $4\left(\frac{\pi}{18} - \frac{\sqrt{3}}{12}\right) = \boxed{\textbf{(B)}\ \frac{2\pi}{9} - \frac{\sqrt{3}}{3}}$

See Also

2010 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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2010 AMC 10B Problems/Problem 17

Contents

Problem

Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37th and 64th, respectively. How many schools are in the city?

$\textbf{(A)}\ 22 \qquad \textbf{(B)}\ 23 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 25 \qquad \textbf{(E)}\ 26$

Solution

Let the n be the number of schools, $3n$ be the number of contestants, and x be Andrea's place. Since the number of participants divided by three is the number of schools, 64 1. Andrea received a higher score than her teammates, so $x \le 36$. Since 36 is the maximum possible median, then $2*36-1=71$ is the maximum possible number of participants. Therefore, $3n \le 71 \Rightarrow n \le \frac{71}{3} = 23\frac{2}{3}$. This yields the compound inequality: $21\frac{1}{3} \le n \le 23\frac{2}{3}$. Since a set with an even number of elements has a median that is the average of the two middle terms, an occurrence that cannot happen in this situation, n cannot be even. $\boxed{\textbf{(B)}\ 23}$ is the only other option.

Solution 2

Let n = the number of schools that participated in the contest.

Then $3n$ students participated in the contest.

Since Andrea's score was the median score of all the students, the number of students in the contest must be odd - we know that no two students can have the same score (from the problem), and we know that if the number of students was even, the middle two scores would have to be the same to get a whole number for the median.

So, we can now write an expression for the median score (Andrea's score) in terms of the number of students who participated in the contest (i.e. $3n$):

$$\frac{3n+1}{2}$$

Also, since we know that this expression must be smaller than 37 (Andrea, whose score was the median, got a better score than Beth and Carla), we can write an inequality and solve it for n:

$$\frac{3n+1}{2} < 37$$

Solving this inequality for n, we get that:

$$n < \frac{73}{3} (\text{which is a little over } 24)$$

So this eliminates answer choices D and E.

But we already know that $3n$ must be odd, implying that n must also be odd! So at this point, the only odd answer choice is $\boxed{\textbf{(B)}\ 23}$, and we are done.

2010 AMC 10B Problems/Problem 18

Problem

Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \dots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3?

$\textbf{(A)}\ \frac{1}{3} \qquad \textbf{(B)}\ \frac{29}{81} \qquad \textbf{(C)}\ \frac{31}{81} \qquad \textbf{(D)}\ \frac{11}{27} \qquad \textbf{(E)}\ \frac{13}{27}$

Solution

First we factor $abc + ab + a$ as $a(bc + b + 1)$, so in order for the number to be divisible by 3, either a is divisible by 3, or $bc + b + 1$ is divisible by 3.

We see that a is divisible by 3 with probability $\frac{1}{3}$. We only need to calculate the probability that $bc + b + 1$ is divisible by 3.

We need $bc + b + 1 \equiv 0 \pmod 3$ or $b(c + 1) \equiv 2 \pmod 3$. Using some modular arithmetic, $b \equiv 2 \pmod 3$ and $c \equiv 0 \pmod 3$ or $b \equiv 1 \pmod 3$ and $c \equiv 1 \pmod 3$. The both cases happen with probability $\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$ so the total probability is $\frac{2}{9}$.

Then the answer is $\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{9} = \frac{13}{27}$ or $\boxed{E}$.

See Also

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2010 AMC 10B Problems/Problem 19

Problem

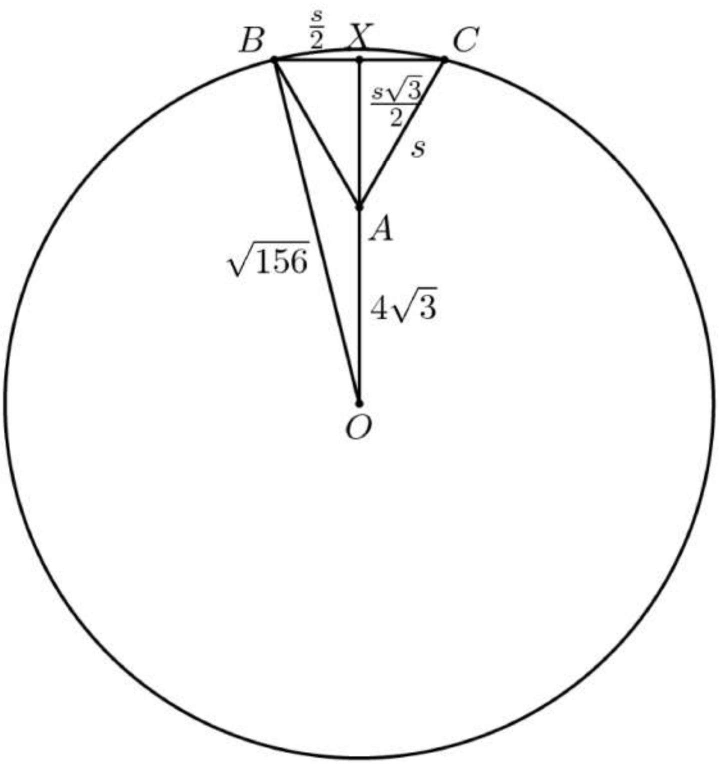
A circle with center O has area 156π. Triangle ABC is equilateral, $\overline{BC}$ is a chord on the circle, $OA = 4\sqrt{3}$, and point O is outside $\triangle ABC$. What is the side length of $\triangle ABC$?

$\textbf{(A)}\ 2\sqrt{3} \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 4\sqrt{3} \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 18$

Solution

The formula for the area of a circle is πr^2 so the radius of this circle is $\sqrt{156}.$

Because $OA = 4\sqrt{3} < \sqrt{156}, A$ must be in the interior of circle $O.$



Let s be the unknown value, the sidelength of the triangle, and let X be the point on BC where $OX \perp BC.$ Since $\triangle ABC$ is equilateral, $BX = \frac{s}{2}$ and $AX = \frac{s\sqrt{3}}{2}.$ We are given $AO = 4\sqrt{3}.$ Use the Pythagorean Theorem and solve for $s.$

$$(\sqrt{156})^2 = \left(\frac{s}{2}\right)^2 + \left(\frac{s\sqrt{3}}{2} + 4\sqrt{3}\right)^2$$
$$156 = \frac{1}{4}s^2 + \frac{3}{4}s^2 + 12s + 48$$
$$0 = s^2 + 12s - 108$$
$$0 = (s-6)(s+18)$$
$$s = \boxed{\textbf{(B)}\ 6}$$

See Also

2010 AMC 10B Problems/Problem 20

Contents

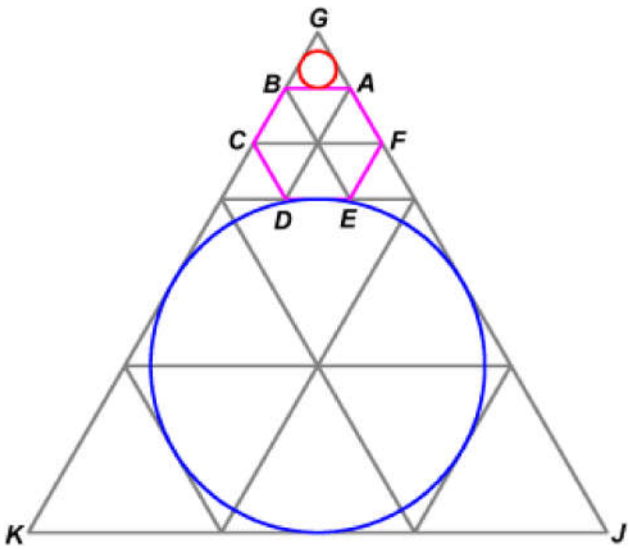
Problem

Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines BC and FA. What is the ratio of the area of the second circle to that of the first circle?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 27 \qquad \textbf{(C)}\ 36 \qquad \textbf{(D)}\ 81 \qquad \textbf{(E)}\ 108$

Solution 1

A good diagram is very helpful.



The first circle is in red, the second in blue. With this diagram, we can see that the first circle is inscribed in equilateral triangle GBA while the second circle is inscribed in GKJ. From this, it's evident that the ratio of the blue area to the red area is equal to the ratio of the areas $\triangle GKJ$ to $\triangle GBA$

Since the ratio of areas is equal to the square of the ratio of lengths, we know our final answer is $\left(\frac{GK}{GB}\right)^2$.

From the diagram, we can see that this is $9^2 = \boxed{\textbf{(D)}\ 81}$

Solution 2

As above, we note that the first circle is inscribed in an equilateral triangle of sidelength 1 (if we assume, WLOG, that the regular hexagon has sidelength 1). The inradius of an equilateral triangle with sidelength 1 is equal to $\frac{\sqrt{3}}{6}$. Therefore, the area of the first circle is $(\frac{\sqrt{3}}{6})^2 \cdot \pi = \frac{\pi}{12}$.

Call the center of the second circle O. Now we drop a perpendicular from O to Circle O's point of tangency with GK and draw another line connecting O to G. Note that because triangle BGA is equilateral, $\angle BGA = 60^\circ$. OG bisects $\angle BGA$, so we have a 30-60-90 triangle.

Call the radius of Circle O $"r"$. $OG = 2r$= height of equilateral triangle + height of regular hexagon + r

The height of an equilateral triangle of sidelength 1 is $\frac{\sqrt{3}}{2}$. The height of an equilateral hexagon of sidelength 1 is $\sqrt{3}$. Therefore, $OG = \frac{\sqrt{3}}{2} + \sqrt{3} + r$.

We can now set up the following equation:

$$\frac{\sqrt{3}}{2} + \sqrt{3} + r = 2r$$
$$\frac{\sqrt{3}}{2} + \sqrt{3} = r$$
$$\frac{3\sqrt{3}}{2} = r$$

The area of Circle O equals $\pi r^2 = \frac{27}{4}\pi$

Therefore, the ratio of the areas is $\frac{\frac{1}{12}}{\frac{27}{4}} = 81$

See Also

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2010 AMC 10B Problems/Problem 21

Problem 21

A palindrome between 1000 and $10,000$ is chosen at random. What is the probability that it is divisible by 7?

$\textbf{(A)}\ \dfrac{1}{10} \qquad \textbf{(B)}\ \dfrac{1}{9} \qquad \textbf{(C)}\ \dfrac{1}{7} \qquad \textbf{(D)}\ \dfrac{1}{6} \qquad \textbf{(E)}\ \dfrac{1}{5}$

Solution

The palindromes can be expressed as: $1000x + 100y + 10y + x$ (since it is a four digit palindrome, it must be of the form $xyyx$, where x and y are integers from $[1,9]$ and $[0,9]$, respectively.)

We simplify this to $1001x + 110y$.

Because the question asks for it to be divisible by 7,

We express it as $1001x + 110y \equiv 0 \pmod{7}$.

Because $1001 \equiv 0 \pmod{7}$,

We can substitute 1001 for 0

We are left with $110y \equiv 0 \pmod{7}$

Since $110 \equiv 5 \pmod{7}$ we can simplify the 110 in the expression to

$5y \equiv 0 \pmod{7}$.

In order for this to be true, $y \equiv 0 \pmod{7}$ must also be true.

Thus we solve:

$y \equiv 0 \pmod{7}$

Which has two solutions: 0 and 7

There are thus two options for y out of the 10, so the answer is $2/10 = \boxed{\textbf{(E)}\ \frac{1}{5}}$

See also

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2010 AMC 10B Problems/Problem 22

Contents

Problem

Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy; the white bag may remain empty. How many arrangements are possible?

$\textbf{(A)}\ 1930 \qquad \textbf{(B)}\ 1931 \qquad \textbf{(C)}\ 1932 \qquad \textbf{(D)}\ 1933 \qquad \textbf{(E)}\ 1934$

Solution 1

We can count the total number of ways to distribute the candies (ignoring the restrictions), and then subtract the overcount to get the answer.

Each candy has three choices; it can go in any of the three bags.

Since there are seven candies, that makes the total distributions $3^7 = 2187$

To find the overcount, we calculate the number of invalid distributions: the red or blue bag is empty.

The number of distributions such that the red bag is empty is equal to 2^7, since it's equivalent to distributing the 7 candies into 2 bags.

We know that the number of distributions with the blue bag is empty will be the same number because of the symmetry, so it's also 2^7.

The case where both the red and the blue bags are empty (all 7 candies are in the white bag) are included in both of the above calculations, and this case has only 1 distribution.

The total overcount is $2^7 + 2^7 - 1 = 2^8 - 1$

The final answer will be
$\text{total} - \text{overcount} = 2187 - (2^8 - 1) = 2187 - 256 + 1 = 1931 + 1 = \boxed{\textbf{(C)}\ 1932}$

Solution 2: Using the Answer Choices

We can first figure out how many ways there are to take two candies from seven distinct candies to place them into the red/blue bags: $7 \cdot 6 = 42$.

Now we can look at the answer choices to find out which ones are divisible by 42, since the total number of combinations must be 42 multiplied by some other number.

Since answers A, B, D, and E are not divisible by 3 (divisor of 42), the answer must be $\boxed{\textbf{(C)}\ 1932}$.

See also

2010 AMC 10B Problems/Problem 23

Contents

Problem

The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 24 \qquad \textbf{(C)}\ 36 \qquad \textbf{(D)}\ 42 \qquad \textbf{(E)}\ 60$

Solution

The upper-left corner must contain the entry 1, and similarly the lower-right corner must contain the entry 9. Consider the entries 2 and 3 -- they may either both lie in the first row, both lie in the first column, or lie in the two squares neighboring 1. By symmetry (which we will take into account by a factor of 2 in the end), we may assume that 2 lies in the cell to the right of 1 and that 3 lies either in the cell to the right of 2 or in the cell below 1:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline &&\\\hline &&\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&\\\hline 3&&\\\hline &&\\\hline\end{array}$$

Similarly, the entries 7 and 8 may either both lie in the last row, or lie in the two squares neighboring 9. This gives the following cases:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline &&\\\hline 7&8&9\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&\\\hline 3&&\\\hline 7&8&9\\\hline\end{array}\times 2 \qquad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline &&7\\\hline &8&9\\\hline\end{array}\times 2 \qquad \begin{array}{|c|c|c|}\hline 1&2&\\\hline 3&&7\\\hline &8&9\\\hline\end{array}\times 2,$$

where the notation $\times 2$ denotes two possible cases, either by switching a row and column or by switching the 7 and 8. Finally, there are respectively 1, 2, 2, 6 ways to complete these four cases. This gives a total of

$$2 \cdot (1 + 2 \times 2 + 2 \times 2 + 2 \times 6) = \boxed{\textbf{(D)}\ 42}$$

possible ways to fill the diagram.

Notes

In fact, there is a general formula (coming from the fields of combinatorics and representation theory) to answer problems of this form; it is known as the hook-length formula (http://en.wikipedia.org/wiki/Young_tableau#Dimension_of_a_representation).

See also

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2010 AMC 10B Problems/Problem 24

Problem

A high school basketball game between the Raiders and Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 31 \qquad \textbf{(C)}\ 32 \qquad \textbf{(D)}\ 33 \qquad \textbf{(E)}\ 34$

Solution

Represent the teams' scores as: (a, an, an^2, an^3) and $(a, a+m, a+2m, a+3m)$

We have $a+an+an^2+an^3 = 4a+6m+1$ Factoring out the a from the left side of the equation, we can get $a(1+n+n^2+n^3) = 4a+6m+1$, or $a(n^4-1)/(n-1) = 4a+6m+1$

Since both are increasing sequences, $n>1$. We can check cases up to $n=4$ because when $n=5$, we get $156a>100$. When

- $n=2, a=[1,6]$
- $n=3, a=[1,2]$
- $n=4, a=1$

Checking each of these cases individually back into the equation $a+an+an^2+an^3 = 4a+6m+1$, we see that only when $a=5$ and $n=2$, we get an integer value for m, which is 9. The original question asks for the first half scores summed, so we must find
$(a)+(an)+(a)+(a+m) = (5)+(10)+(5)+(5+9) = \boxed{\textbf{(E)}\ 34}$

See also

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2010 AMC 10B Problems/Problem 25

Contents

Problem

Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a, \text{ and}$$
$$P(2) = P(4) = P(6) = P(8) = -a.$$

What is the smallest possible value of a?

$\textbf{(A)}\ 105 \qquad \textbf{(B)}\ 315 \qquad \textbf{(C)}\ 945 \qquad \textbf{(D)}\ 7! \qquad \textbf{(E)}\ 8!$

Solution

We observe that because $P(1) = P(3) = P(5) = P(7) = a$, if we define a new polynomial $R(x)$ such that $R(x) = P(x) - a$, $R(x)$ has roots when $P(x) = a$; namely, when $x = 1, 3, 5, 7$.

Thus since $R(x)$ has roots when $x = 1, 3, 5, 7$, we can factor the product $(x-1)(x-3)(x-5)(x-7)$ out of $R(x)$ to obtain a new polynomial $Q(x)$ such that $(x-1)(x-3)(x-5)(x-7)(Q(x)) = R(x) = P(x) - a$.

Then, plugging in values of $2, 4, 6, 8$, we get

$$P(2) - a = (2-1)(2-3)(2-5)(2-7)Q(2) = -15Q(2) = -2a$$

$$P(4) - a = (4-1)(4-3)(4-5)(4-7)Q(4) = 9Q(4) = -2a$$

$$P(6) - a = (6-1)(6-3)(6-5)(6-7)Q(6) = -15Q(6) = -2a$$

$$P(8) - a = (8-1)(8-3)(8-5)(8-7)Q(8) = 105Q(8) = -2a$$

$-2a = -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8)$. Thus, the least value of a must be the $lcm(15, 9, 15, 105)$. Solving, we receive 315, so our answer is $\boxed{\textbf{(B)}\ 315}$.

Critique

The above solution is incomplete. What is really proven is that 315 is a factor of a, if such an a exists. That only rules out answer A.

To prove that the answer is correct, one could exhibit a polynomium that satisfies the requirements with $a = 315$. Here's one:
$$P(x) = -8x^7 + 252x^6 - 3248x^5 + 22050x^4 - 84392x^3 + 179928x^2 - 194592x + 80325.$$

You get that from the 8×8 matrix $M_{i,j} = i^{j-1}$ and $y^T = (315, -315, \ldots, 315, -315)$ and computing $c = M^{-1}y$ which comes out as the all-integer coefficients above.

Critique of Critique

First of all, the solution shows that a is a multiple of 315, not a factor of 315. Many people confuse the usage of the words 'factor' and 'multiple'. Secondly, even if a is a multiple of 315, it does not mean that you can instantly get that the answer is 315 because we need to know that $a = 315$ is possible. After all, a is also a multiple of 1, but 1 is definitely not the smallest possible number.

To complete the solution, we can let $a = 315$, and then try to find $Q(x)$. We know from the above calculation that $Q(2) = 42, Q(4) = -70, Q(6) = 42$, and $Q(8) = -6$. Then we can let $Q(x) = T(x)(x-2)(x-6) + 42$, getting $T(4) = 28, T(8) = -4$. Let $T(x) = L(x)(x-8) - 4$, then $L(4) = -8$. Therefore, it is possible to choose $T(x) = -8(x-8) - 4 = -8x + 60$, so the goal is accomplished. As a reference, the polynomial we get is

$$P(x) = (x-1)(x-3)(x-5)(x-7)((-8x+60)(x-2)(x-6)+42)+315$$

$$= -8x^7+252x^6-3248x^5+22050x^4-84392x^3+179928x^2-194592x+80325$$

Critique of Critique of Critique

The problem states the "least value" of a, so it is not needed to add the extra steps.

Critique of Critique of Critique of Critique

It is still necessary to show that the minimum is achievable. For example $x^2 + 1 > 0$, but 0 is not the least value of $x^2 + 1$

See also

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