2011 AMC 10A Problems/Problem 1

Problem 1

A cell phone plan costs $\$20$ each month, plus $5¢$ per text message sent, plus $10¢$ for each minute used over 30 hours. In January Michelle sent 100 text messages and talked for 30.5 hours. How much did she have to pay?

$\textbf{(A)}\ \$24.00 \qquad \textbf{(B)}\ \$24.50 \qquad \textbf{(C)}\ \$25.50 \qquad \textbf{(D)}\ \$28.00 \qquad \textbf{(E)}\ \$30.00$

Solution

$30.5-30=.5$ hours which is 30 minutes. $30*10=300$ cents which is 3 dollars. $100*.05=5$ dollars. So finally $20+3+5=28$ dollars. $\longrightarrow \boxed{\textbf{D}}$

See Also

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2011 AMC 10A Problems/Problem 2

Problem 2

A small bottle of shampoo can hold 35 milliliters of shampoo, whereas a large bottle can hold 500 milliliters of shampoo. Jasmine wants to buy the minimum number of small bottles necessary to completely fill a large bottle. How many bottles must she buy?

$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 12 \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ 14 \qquad \textbf{(E)}\ 15$

Solution

You want to find the minimum number of small bottles: so you do $\frac{500}{35} \approx 14.3$ which you round (up for our purposes) to 15.

The answer is $\boxed{15 \textbf{(E)}}$.

See Also

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2011 AMC 10A Problems/Problem 3

Problem 3

Suppose [a b] denotes the average of a and b, and {a b c} denotes the average of a, b, and c. What is {{1 1 0} [0 1] 0}?

$\textbf{(A)}\ \frac{2}{9} \qquad \textbf{(B)}\ \frac{5}{18} \qquad \textbf{(C)}\ \frac{1}{3} \qquad \textbf{(D)}\ \frac{7}{18} \qquad \textbf{(E)}\ \frac{2}{3}$

Solution

Average 1, 1, and 0 to get $\frac{2}{3}$. Average 0, and 1, to get $\frac{1}{2}$. Average $\frac{2}{3}$, $\frac{1}{2}$, and 0. to get $\boxed{\textbf{(D)}\ \frac{7}{18}}$

See Also

2011 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2011))	
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2011 AMC 10A Problems/Problem 4

Let X and Y be the following sums of arithmetic sequences:

$$\begin{aligned} X &= 10+12+14+\cdots+100, \\ Y &= 12+14+16+\cdots+102. \end{aligned}$$

What is the value of Y - X?

$\textbf{(A)}\ 92\qquad\textbf{(B)}\ 98\qquad\textbf{(C)}\ 100\qquad\textbf{(D)}\ 102\qquad\textbf{(E)}\ 112$

Solution 1

We see that both sequences have equal numbers of terms, so reformat the sequence to look like:

$$Y = 12+14+\cdots+100+102$$
$$X = 10+12+14+\cdots+100$$

From here it is obvious that Y - X = 102 - 10 = $\boxed{92\ (\textbf{A})}$.

Solution 2

We see that every number in Y's sequence is two more than every corresponding number in X's sequence. Since there are 46 numbers in each sequence, the difference must be: $46*2=\boxed{92}$

See Also

2011 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2011))	
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2011 AMC 10A Problems/Problem 5

Problem 5

At an elementary school, the students in third grade, fourth grade, and fifth grade run an average of 12, 15, and 10 minutes per day, respectively. There are twice as many third graders as fourth graders, and twice as many fourth graders as fifth graders. What is the average number of minutes run per day by these students?

$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ \frac{37}{3} \qquad \textbf{(C)}\ \frac{88}{7} \qquad \textbf{(D)}\ 13 \qquad \textbf{(E)}\ 14$

Solution

Let there be x fifth graders. It follows that there are $2x$ fourth graders and $4x$ third graders. We have $\frac{(1x)(10)+(2x)(15)+(4x)(12)}{1x+2x+4x} = \boxed{\textbf{(C)}\frac{88}{7}}$.

See Also

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2011 AMC 10A Problems/Problem 6

Problem 6

Set A has 20 elements, and set B has 15 elements. What is the smallest possible number of elements in $A \cup B$?

$\textbf{(A)} 5 \qquad \textbf{(B)}\ 15 \qquad \textbf{(C)}\ 20 \qquad \textbf{(D)}\ 35 \qquad \textbf{(E)}\ 300$

Solution

$A \cup B$ will be smallest if B is completely contained in A, in which case all the elements in B would be counted for in A. So the total would be the number of elements in A, which is $\boxed{20 \ \textbf{(C)}}$.

See Also

2011 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2011))	
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2011 AMC 10A Problems/Problem 7

Problem 7

Which of the following equations does NOT have a solution?

$\text{(A)}\; (x+7)^2 = 0$

$\text{(B)}\; |-3x|+5=0$

$\text{(C)}\; \sqrt{-x}-2=0$

$\text{(D)}\; \sqrt{x}-8=0$

$\text{(E)}\; |-3x|-4=0$

Solution

$|-3x|+5=0$ has no solution because absolute values output positives and this equation implies that the absolute value could output a negative.

Further: $(x+7)^2=0$ is true for $x=-7$

$\sqrt{-x}-2=0$ is true for $x=-4$

$\sqrt{x}-8=0$ is true for $x=64$

$|-3x|-4=0$ is true for $x=\frac{4}{3},-\frac{4}{3}$

Therefore, the answer is $\boxed{\text{(B)}}$.

See Also

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2011 AMC 10A Problems/Problem 8

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Problem 8

Last summer 30% of the birds living on Town Lake were geese, 25% were swans, 10% were herons, and 35% were ducks. What percent of the birds that were not swans were geese?

$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 50 \qquad \textbf{(E)}\ 60$

Solutions

Solution 1

75% of the total birds were not swans. Out of that 75%, there was $30\%/75\% = \boxed{40\% (C)}$ of the birds that were not swans that were geese.

Solution 2

Suppose there were 100 birds in total living on Town Lake, then 30 were geese, 25 were swans, 10 were herons, and 35 were ducks. $100-25=75$ of the birds are not swans and 30 of these are geese, so the answer is $\frac{30}{75}\times 100 = \boxed{40\% (C)}$.

See Also

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2011 AMC 10A Problems/Problem 9

Problem 9

A rectangular region is bounded by the graphs of the equations $y = a, y = -b, x = -c,$ and $x = d,$ where $a, b, c,$ and d are all positive numbers. Which of the following represents the area of this region?

$\textbf{(A)}\ ac+ad+bc+bd \qquad \textbf{(B)}\ ac-ad+bc-bd \qquad \textbf{(C)}\ ac+ad-bc-bd \qquad \textbf{(D)}\ -ac-ad+bc+bd \qquad \textbf{(E)}\ ac-ad-bc+bd$

Solution

We have a rectangle of side lengths $a-(-b)=a+b$ and $d-(-c)=c+d.$ Thus the area of this rectangle is $(a+b)(c+d)=\boxed{\textbf{(A)}\ ac+ad+bc+bd}$.

See Also

2011 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2011))	
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2011 AMC 10A Problems/Problem 10

Problem 10

A majority of the 30 students in Ms. Demeanor's class bought pencils at the school bookstore. Each of these students bought the same number of pencils, and this number was greater than 1. The cost of a pencil in cents was greater than the number of pencils each student bought, and the total cost of all the pencils was $\$17.71$. What was the cost of a pencil in cents?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 11 \qquad \textbf{(C)}\ 17 \qquad \textbf{(D)}\ 23 \qquad \textbf{(E)}\ 77$

Solution

Let $x > 15$ be the number of students at the bookstore, $p > 1$ be the number of pencils each student bought, and $c > p$ be the price of one pencil. We see that $xcp = 1771 = 7 \cdot 11 \cdot 23$. Then we must have $x = 23, c = 11, p = 7$ by the original conditions and the answer is $c = \boxed{11 \ \mathbf{(B)}}$.

See Also

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2011 AMC 10A Problems/Problem 11

Problem 11

Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on AB with $AE = 7 \cdot EB$. What is the ratio of the area of $EFGH$ to the area of $ABCD$?

$\text{(A)}\ \frac{49}{64} \qquad \text{(B)}\ \frac{25}{32} \qquad \text{(C)}\ \frac78 \qquad \text{(D)}\ \frac{5\sqrt{2}}{8} \qquad \text{(E)}\ \frac{\sqrt{14}}{4}$

Solution

Let $8s$ be the length of the sides of square $ABCD$, then the length of one of the sides of square $EFGH$ is $\sqrt{(7s)^2+s^2}=\sqrt{50s^2}$, and hence the ratio in the areas is $\frac{\sqrt{50s^2}^2}{(8s)^2}=\frac{50}{64}=\boxed{\frac{25}{32} \ \mathbf{(B)}}$.

See Also

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Category: Introductory Geometry Problems

2011 AMC 10A Problems/Problem 12

Problem 12

The players on a basketball team made some three-point shots, some two-point shots, and some one-point free throws. They scored as many points with two-point shots as with three-point shots. Their number of successful free throws was one more than their number of successful two-point shots. The team's total score was 61 points. How many free throws did they make?

$\textbf{(A)}\ 13 \qquad \textbf{(B)}\ 14 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 17$

Solution

Suppose there were x three-point shots, y two-point shots, and z one-point shots. Then we get the following system of equations:

$$\begin{aligned} 3x &= 2y && (1) \\ z &= y+1 && (2) \\ 3x+2y+z &= 61 && (3) \end{aligned}$$

The value we are looking for is z, which is easily found to be $z = \boxed{13\ \textbf{(A)}}$.

See Also

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2011 AMC 10A Problems/Problem 13

Problem 13

How many even integers are there between 200 and 700 whose digits are all different and come from the set $\{1,2,5,7,8,9\}$?

$\textbf{(A)}\,12\qquad\textbf{(B)}\,20\qquad\textbf{(C)}\,72\qquad\textbf{(D)}\,120\qquad\textbf{(E)}\,200$

Solution

We split up into cases of the hundreds digits being 2 or 5. If the hundred digits is 2, then the units digits must be 8 in order for the number to be even and then there are 4 remaining choices $(1,5,7,9)$ for the tens digit, giving $1\times1\times4=4$ possibilities. Similarly, there are $1\times2\times4=8$ possibilities for the 5 case, giving a total of $\boxed{4+8=12\ \textbf{(A)}}$ possibilities.

See Also

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2011 AMC 10A Problems/Problem 14

Problem 14

A pair of standard 6-sided fair dice is rolled once. The sum of the numbers rolled determines the diameter of a circle. What is the probability that the numerical value of the area of the circle is less than the numerical value of the circle's circumference?

$\textbf{(A)}\ \frac{1}{36} \qquad \textbf{(B)}\ \frac{1}{12} \qquad \textbf{(C)}\ \frac{1}{6} \qquad \textbf{(D)}\ \frac{1}{4} \qquad \textbf{(E)}\ \frac{5}{18}$

Solution

We want the area, πr^2, to be less than the circumference, $2\pi r$:

$$\pi r^2 < 2\pi r$$
$$r < 2$$

If $r < 2$ then the dice must show $(1,1), (1,2), (2,1)$ which are 3 choices out of a total possible of $6 \times 6 = 36$, so the probability is $\frac{3}{36} = \boxed{\frac{1}{12} \ \mathbf{(B)}}$.

See Also

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2011 AMC 10A Problems/Problem 15

Problem 15

Roy bought a new battery-gasoline hybrid car. On a trip the car ran exclusively on its battery for the first 40 miles, then ran exclusively on gasoline for the rest of the trip, using gasoline at a rate of 0.02 gallons per mile. On the whole trip he averaged 55 miles per gallon. How long was the trip in miles?

(A) 140 (B) 240 (C) 440 (D) 640 (E) 840

Solution

We know that $\frac{\text{total miles}}{\text{total gas}} = 55$. Let x be the distance the car traveled during the time it ran on gasoline, then the amount of gas used is $0.02x$. The total distance traveled is $40 + x$, so we get $\frac{40+x}{0.02x} = 55$. Solving this equation, we get $x = 400$, so the total distance is $400 + 40 = \boxed{440 \ \textbf{(C)}}$.

See Also

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2011 AMC 10A Problems/Problem 16

Problem 16

Which of the following is equal to $\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}$?

$\text{(A)}\,3\sqrt2 \qquad\text{(B)}\,2\sqrt6 \qquad\text{(C)}\,\frac{7\sqrt2}{2} \qquad\text{(D)}\,3\sqrt3 \qquad\text{(E)}\,6$

Solution

We find the answer by squaring, then square rooting the expression.

$$\begin{aligned}
&\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}\\
&=\sqrt{\left(\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}\right)^2}\\
&=\sqrt{9-6\sqrt{2}+2\sqrt{(9-6\sqrt{2})(9+6\sqrt{2})}+9+6\sqrt{2}}\\
&=\sqrt{18+2\sqrt{(9-6\sqrt{2})(9+6\sqrt{2})}}\\
&=\sqrt{18+2\sqrt{9^2-(6\sqrt{2})^2}}\\
&=\sqrt{18+2\sqrt{81-72}}\\
&=\sqrt{18+2\sqrt{9}}\\
&=\sqrt{18+6}\\
&=\sqrt{24}\\
&=\boxed{2\sqrt{6}\ \textbf{(B)}}
\end{aligned}$$

See Also

2011 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2011))	
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2011 AMC 10A Problems/Problem 17

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Problem 17

In the eight-term sequence $A,B,C,D,E,F,G,H,$ the value of C is 5 and the sum of any three consecutive terms is 30. What is $A+H$?

$\textbf{(A)}\,17 \qquad\textbf{(B)}\,18 \qquad\textbf{(C)}\,25 \qquad\textbf{(D)}\,26 \qquad\textbf{(E)}\,43$

Solution

We consider the sum $A+B+C+D+E+F+G+H$ and use the fact that $C=5$, and hence $A+B=25$.

$$A+B+C+D+E+F+G+H=A+(B+C+D)+(E+F+G)+H=A+30+30+H=A+H+60$$
$$A+B+C+D+E+F+G+H=(A+B)+(C+D+E)+(F+G+H)=25+30+30=85$$

Equating the two values we get for the sum, we get the answer $A+H+60=85 \implies A+H=\boxed{25\ (\mathbf{C})}$.

Solution 2

We see that $A+B+C=30$, and by substituting the given $C=5$, we find that $A+B=25$. Similarly, $B+D=25$ and $D+E=25$.

$$\begin{aligned}
&(A+B)-(B+D)=A-D=0\\
&A=D\\
&(B+D)-(D+E)=B-E=0\\
&B=E\\
&A,B,5,A,B,5,G,H
\end{aligned}$$

Similarly, $G=A$ and $H=B$, giving us $A,B,5,A,B,5,A,B$. Since $H=B$, $A+H=A+B=\boxed{25\ (\mathbf{C})}$.

See Also

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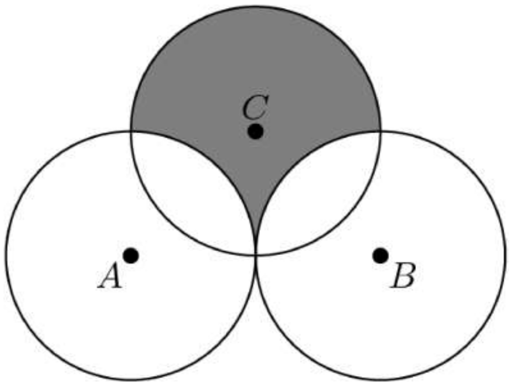
2011 AMC 10A Problems/Problem 18

Problem 18

Circles $A, B,$ and C each have radius 1. Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}$. What is the area inside Circle C but outside circle A and circle B ?



$\textbf{(A)}\ 3 - \frac{\pi}{2} \qquad \textbf{(B)}\ \frac{\pi}{2} \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ \frac{3\pi}{4} \qquad \textbf{(B)}\ 1+\frac{\pi}{2}$

Solution

Draw a rectangle with vertices at the centers of A and B and the intersection of A, C and B, C. Then, we can compute the shaded area as the area of half of C plus the area of the rectangle minus the area of the two sectors created by A and B. This is $\frac{\pi(1)^2}{2}+(2)(1)-2\cdot\frac{\pi(1)^2}{4}=\boxed{\textbf{(C)} 2}$.

See Also

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Category: Introductory Geometry Problems

2011 AMC 10A Problems/Problem 19

Problem 19

In 1991 the population of a town was a perfect square. Ten years later, after an increase of 150 people, the population was 9 more than a perfect square. Now, in 2011, with an increase of another 150 people, the population is once again a perfect square. Which of the following is closest to the percent growth of the town's population during this twenty-year period?

$\textbf{(A)}\ 42 \qquad\textbf{(B)}\ 47 \qquad\textbf{(C)}\ 52 \qquad\textbf{(D)}\ 57 \qquad\textbf{(E)}\ 62$

Solution

Let the population of the town in 1991 be p^2. Let the population in 2001 be q^2+9. Let the population in 2011 be r^2. It follows that $p^2+150=q^2+9$. Rearrange this equation to get $141=q^2-p^2=(q-p)(q+p)$. Since q and p are both positive integers with $q>p$, $(q-p)$ and $(q+p)$ also must be, and thus, they are both factors of 141. We have two choices for pairs of factors of 141: 1 and 141, and 3 and 47. Assuming the former pair, since $(q-p)$ must be less than $(q+p)$, $q-p=1$ and $q+p=141$. Solve to get $p=70, q=71$. Since p^2+300 is not a perfect square, this is not the correct pair. Solve for the other pair to get $p=22, q=25$. This time, $p^2+300=22^2+300=784=28^2$. This is the correct pair. Now, we find the percent increase from $22^2=484$ to $28^2=784$. Since the increase is 300, the percent increase is $\frac{300}{484}\times100\% \approx \boxed{\textbf{(E)}\ 62\%}$.

See Also

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2011 AMC 10A Problems/Problem 20

Problem 20

Two points on the circumference of a circle of radius r are selected independently and at random. From each point a chord of length r is drawn in a clockwise direction. What is the probability that the two chords intersect?

$\textbf{(A)}\ \frac{1}{6}\qquad\textbf{(B)}\ \frac{1}{5}\qquad\textbf{(C)}\ \frac{1}{4}\qquad\textbf{(D)}\ \frac{1}{3}\qquad\textbf{(E)}\ \frac{1}{2}$

Solution

Fix a point A from which we draw a clockwise chord. In order for the clockwise chord from another point B to intersect that of point A, A and B must be no more than r units apart. By drawing the circle, we quickly see that B can be on $\frac{120}{360}=\boxed{\frac{1}{3}\ \textbf{(D)}}$ of the perimeter of the circle. (Imagine a regular hexagon inscribed in the circle)

See Also

2011 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2011))	
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Category: Introductory Geometry Problems

2011 AMC 10A Problems/Problem 21

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Problem 21

Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins. A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the remaining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine?

$\textbf{(A)}\ \frac{7}{11}\qquad\textbf{(B)}\ \frac{9}{13}\qquad\textbf{(C)}\ \frac{11}{15}\qquad\textbf{(D)}\ \frac{15}{19}\qquad\textbf{(E)}\ \frac{15}{16}$

Solution 1

Note that we are trying to find the conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$ where A is the 4 coins being genuine and B is the sum of the weight of the coins being equal. The only possibilities for B are (g is abbreviated for genuine and c is abbreviated for counterfeit, and the pairs are the first and last two in the quadruple) $(g,g,g,g),(g,c,g,c),(g,c,c,g),(c,g,g,c),(c,g,c,g)$. We see that $A\cap B$ happens with probability $\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}=\frac{1}{3}$, and B happens with probability $\frac{1}{3}+4\times\left(\frac{8}{10}\times\frac{2}{9}\times\frac{7}{8}\times\frac{1}{7}\right)=\frac{19}{45}$, hence $P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{3}}{\frac{19}{45}}=\boxed{\frac{15}{19}\ \textbf{(D)}}$.

Solution 2

If we pick 4 indistinguishable real coins from the set of 8 real coins, there are $\binom{8}{4}$ ways to pick the coins. If we then place the coins in four distinguishable slots on the scale, there are $4!$ ways to arrange them, giving $4!\cdot\binom{8}{4}$ ways to choose and place 8 real coins. This gives 1680 desirable combinations.

If we pick 2 real coins and 2 fake coins, there are $\binom{8}{2}\binom{2}{2}$ ways to choose the coins. There are 4 choices for the first slot on the left side of the scale. Whichever type of coin is placed in that first slot, there are 2 choices for the second slot on the left side of the scale, since it must be of the opposite type of coin. There are 2 choices for the first slot on the right side of the scale, and only 1 choice for the last slot on the right side.

Thus, there are $4\cdot 2\cdot 2\cdot 1=16$ ways to arrange the coins, and $\binom{8}{2}\binom{2}{2}=28$ sets of possible coins, for a total of $16\cdot 28=448$ combinations that are legal, yet undesirable.

The overall probability is thus $\frac{1680}{1680+448}=\boxed{\frac{15}{19}\ \textbf{(D)}}$.

Note that in this solution, all four slots on the scale are deemed to be distinguishable. In essence, this "overcounts" all numbers by a factor of 8, since you can switch the coins on the left side, you can switch the coins on the right side, or you can switch sides of the scale. But since all numbers are increased 8-fold, it will cancel out when calculating the probability.

Solution 3

Place the two coins from the first pair in a line followed by the two coins from the second pair followed by the six left-over coins. We can do that in $\binom{10}{2} = 45$ different ways.

We need to exclude those cases where the weight shows a difference. There are two cases where a pair has both counterfeit coins and there are $4 \cdot 6$ cases where one counterfeit coin is chosen and one is in the left-over. That leaves $45 - 2 - 24 = 19$ cases.

Of these, $\binom{6}{2} = 15$ cases has both counterfeit coins in the left-over. The probability of having chosen four genuine coins therefore is $\boxed{\frac{15}{19}\ \textbf{(D)}}$.

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Category: Introductory Combinatorics Problems

2011 AMC 10A Problems/Problem 22

Problem 22

Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

$\textbf{(A)}\ 2520 \qquad \textbf{(B)}\ 2880 \qquad \textbf{(C)}\ 3120 \qquad \textbf{(D)}\ 3250 \qquad \textbf{(E)}\ 3750$

Solution

Let vertex A be any vertex, then vertex B be one of the diagonal vertices to A, C be one of the diagonal vertices to B, and so on. We consider cases for this problem.

In the case that C has the same color as A, D has a different color from A and so E has a different color from A and D. In this case, A has 6 choices, B has 5 choices (any color but the color of A), C has 1 choice, D has 5 choices, and E has 4 choices, resulting in a possible of $6\cdot5\cdot1\cdot5\cdot4=600$ combinations.

In the case that C has a different color from A and D has a different color from A, A has 6 choices, B has 5 choices, C has 4 choices (since A and B necessarily have different colors), D has 4 choices, and E has 4 choices, resulting in a possible of $6\cdot5\cdot4\cdot4\cdot4=1920$ combinations.

In the case that C has a different color from A and D has the same color as A, A has 6 choices, B has 5 choices, C has 4 choices, D has 1 choice, and E has 5 choices, resulting in a possible of $6\cdot5\cdot4\cdot1\cdot5=600$ combinations.

Adding all those combinations up, we get $600+1920+600=\boxed{3120 \ \textbf{(C)}}$.

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Category: Intermediate Combinatorics Problems

2011 AMC 10A Problems/Problem 23

Problem

Seven students count from 1 to 1000 as follows:

•Alice says all the numbers, except she skips the middle number in each consecutive group of three numbers. That is, Alice says 1, 3, 4, 6, 7, 9, . . ., 997, 999, 1000.

•Barbara says all of the numbers that Alice doesn't say, except she also skips the middle number in each consecutive group of three numbers.

•Candice says all of the numbers that neither Alice nor Barbara says, except she also skips the middle number in each consecutive group of three numbers.

•Debbie, Eliza, and Fatima say all of the numbers that none of the students with the first names beginning before theirs in the alphabet say, except each also skips the middle number in each of her consecutive groups of three numbers.

•Finally, George says the only number that no one else says.

What number does George say?

$\textbf{(A)}\ 37\qquad\textbf{(B)}\ 242\qquad\textbf{(C)}\ 365\qquad\textbf{(D)}\ 728\qquad\textbf{(E)}\ 998$

Solution

First look at the numbers Alice says. $1,3,4,6,7,9\cdots$ skipping every number that is congruent to $2 \pmod 3$. Thus, Barbara says those numbers EXCEPT every second - being $2+3^1\equiv 5 \pmod{3^2=9}$. So Barbara skips every number congruent to $5 \pmod 9$. We continue and see:

Alice skips $2 \pmod 3$, Barbara skips $5 \pmod 9$, Candice skips $14 \pmod{27}$, Debbie skips $41 \pmod{81}$, Eliza skips $122 \pmod{243}$, and Fatima skips $365 \pmod{729}$.

Since the only number congruent to $365 \pmod{729}$ and less than $1,000$ is 365, the correct answer is $\boxed{365\ \textbf{(C)}}$.

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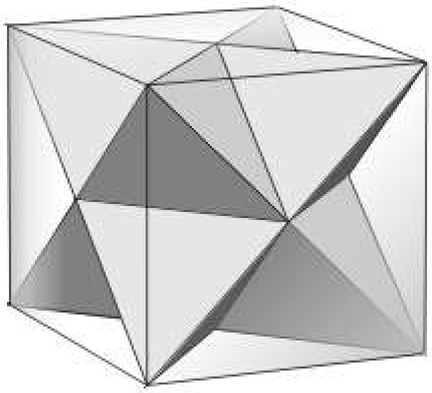
Problem 24

Two distinct regular tetrahedra have all their vertices among the vertices of the same unit cube. What is the volume of the region formed by the intersection of the tetrahedra?

$\textbf{(A)}\ \frac{1}{12}\qquad\textbf{(B)}\ \frac{\sqrt{2}}{12}\qquad\textbf{(C)}\ \frac{\sqrt{3}}{12}\qquad\textbf{(D)}\ \frac{1}{6}\qquad\textbf{(E)}\ \frac{\sqrt{2}}{6}$

Solution

The two tetrahedra look somewhat like this.



Solution 1

A regular unit tetrahedron can be split into eight tetrahedra that have lengths of $\frac{1}{2}$. The volume of a regular tetrahedron can be found using base area and height:

For a tetrahedron of side length 1, its base area is $\frac{\sqrt{3}}{4}$, and its height can be found using Pythagoras' Theorem.

Its height is $\sqrt{1^2-\left(\frac{\sqrt{3}}{3}\right)^2}=\frac{\sqrt{2}}{\sqrt{3}}$. Its volume is $\frac{1}{3}\times\frac{\sqrt{3}}{4}\times\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{2}}{12}$.

The tetrahedron actually has side length $\sqrt{2}$, so the actual volume is $\frac{\sqrt{2}}{12}\times\sqrt{2}^3=\frac{1}{3}$.

On the eight small tetrahedra, the four tetrahedra on the corners of the large tetrahedra are not inside the other large tetrahedra. Thus, $\frac{4}{8}=\frac{1}{2}$ of the large tetrahedra will not be inside the other large tetrahedra.

The intersection of the two tetrahedra is thus $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}=\boxed{\textbf{(D)}}$.

Solution 2

The intersection of the two tetrahedra is an octahedron that has points touching the center of each face of the original cube. We can split the octahedron into two square pyramids, and from there we can find the volume of each pyramid. The sides of the square face of the pyramid will have lengths of $\frac{\sqrt{2}}{2}$, so the volume of both pyramids (the whole octahedra) will be $\left(\frac{\sqrt{2}}{2}\right)^2 \times 2 \times \frac{1}{2} \times \frac{1}{3} = 1/6 = \boxed{\text{(D)}}$.

See Also

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Category: Introductory Geometry Problems

2011 AMC 10A Problems/Problem 25

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Problem 25

Let R be a square region and $n \geq 4$ an integer. A point X in the interior of R is called $n\text{-}ray$ partitional if there are n rays emanating from X that divide R into n triangles of equal area. How many points are 100-ray partitional but not 60-ray partitional?

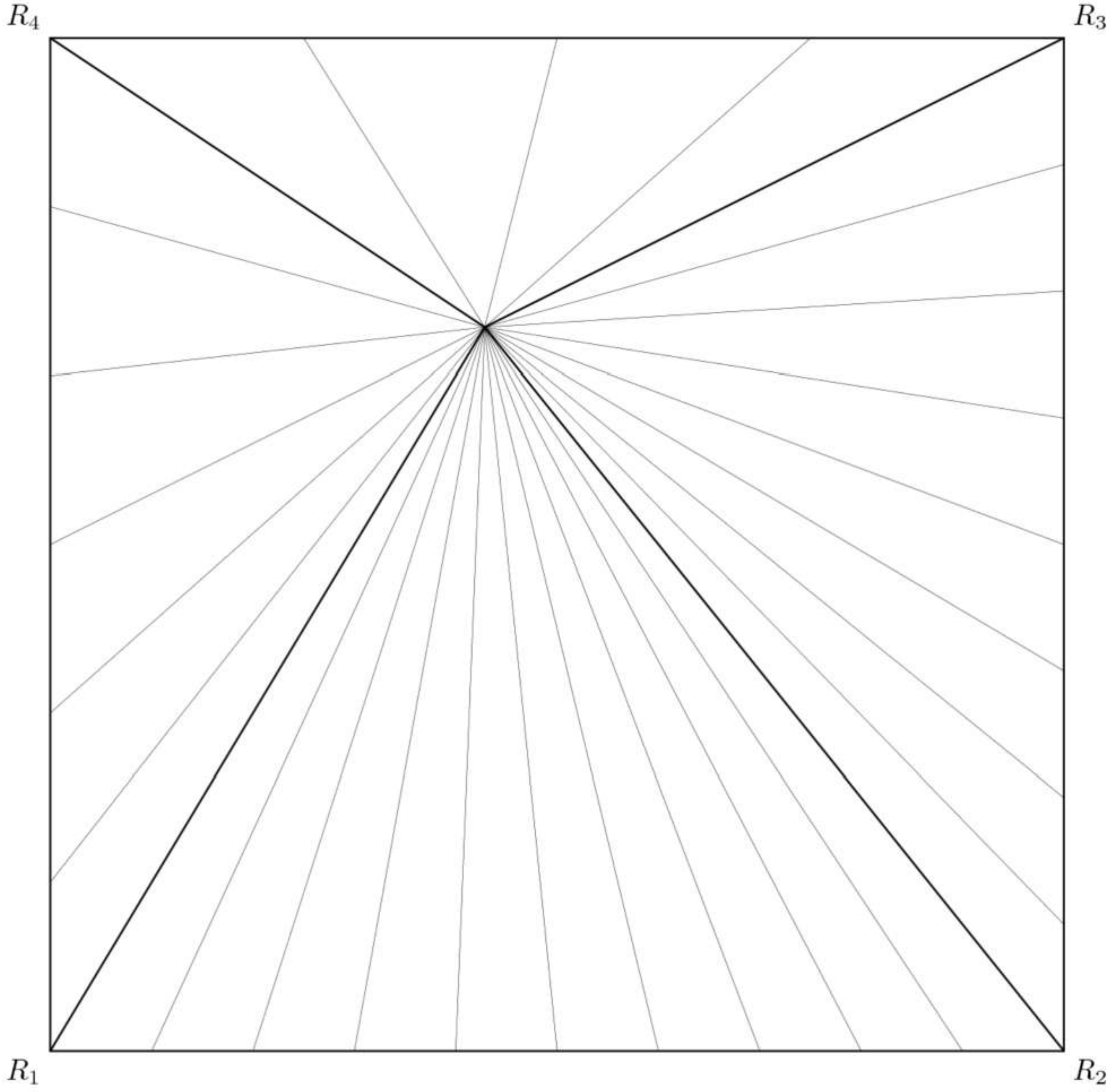
$\textbf{(A)}\ 1500 \qquad \textbf{(B)}\ 1560 \qquad \textbf{(C)}\ 2320 \qquad \textbf{(D)}\ 2480 \qquad \textbf{(E)}\ 2500$

Solution 1

First, notice that there must be four rays emanating from X that intersect the four corners of the square region. Depending on the location of X, the number of rays distributed among these four triangular sectors will vary. We start by finding the corner-most point that is 100-ray partitional (let this point be the bottom-left-most point). We first draw the four rays that intersect the vertices. At this point, the triangular sectors with bases as the sides of the square that the point is closest to both do not have rays dividing their areas. Therefore, their heights are equivalent since their areas are equal. The remaining 96 rays are divided among the other two triangular sectors, each sector with 48 rays, thus dividing these two sectors into 49 triangles of equal areas. Let the distance from this corner point to the closest side be a and the side of the square be s. From this, we get the equation $\frac{a\times s}{2}=\frac{(s-a)\times s}{2}\times\frac1{49}$. Solve for a to get $a=\frac s{50}$. Therefore, point X is $\frac1{50}$ of the side length away from the two sides it is closest to. By moving X $\frac s{50}$ to the right, we also move one ray from the right sector to the left sector, which determines another 100-ray partitional point. We can continue moving X right and up to derive the set of points that are 100-ray partitional. In the end, we get a square grid of points each $\frac s{50}$ apart from one another. Since this grid ranges from a distance of $\frac s{50}$ from one side to $\frac{49s}{50}$ from the same side, we have a 49×49 grid, a total of 2401 100-ray partitional points. To find the overlap from the 60-ray partitional, we must find the distance from the corner-most 60-ray partitional point to the sides closest to it. Since the 100-ray partitional points form a 49×49 grid, each point $\frac s{50}$ apart from each other, we can deduce that the 60-ray partitional points form a 29×29 grid, each point $\frac s{30}$ apart from each other. To find the overlap points, we must find the common divisors of 30 and 50 which are $1, 2, 5,$ and 10. Therefore, the overlapping points will form grids with points s, $\frac s{2}$, $\frac s{5}$, and $\frac s{10}$ away from each other respectively. Since the grid with points $\frac s{10}$ away from each other includes the other points, we can disregard the other grids. The total overlapping set of points is a 9×9 grid, which has 81 points. Subtract 81 from 2401 to get $2401-81=\boxed{\textbf{(C)}\ 2320}$.

Solution 2

We may assume that the square R has coordinates $R_1(0,0),R_2(1,0),R_3(1,1),R_4(0,1)$. Suppose that $X=(x,y)$ is n-ray partitional.



Example: $X = (\frac{3}{7}, \frac{5}{7})$ is 28-ray partitional.

By definition, there exist n rays from X which divide R into n triangles of equal area, and 4 of these rays intersect the vertices of R. Let a, b, c, d be the number of triangles that share a side with $\overline{R_1R_2}, \overline{R_2R_3}, \overline{R_3R_4}, \overline{R_4R_1}$, respectively. Then

$$a + b + c + d = n \; .$$

Let the common area of the triangles be A. Each of the triangles that share a side with $\overline{R_1R_2}$ have common height y; since they have the same area, they must have the same base $\frac{1}{a}$. Hence $A = \frac{1}{2} \cdot \frac{1}{a} \cdot y$. Similarly,

$$A = \frac{y}{2a} = \frac{1-x}{2b} = \frac{1-y}{2c} = \frac{x}{2d} \; .$$

Substituting gives

$$n = \frac{y}{2A} + \frac{1-x}{2A} + \frac{1-y}{2A} + \frac{x}{2A} = \frac{1}{A}$$

and

$$a = \frac{n}{2}y \; , \; b = \frac{n}{2}(1-x) \; , \; c = \frac{n}{2}(1-y) \; , \; d = \frac{n}{2}x \; .$$

Since $x = \frac{2d}{n}$ and $y = \frac{2a}{n}$ and a, d, n are integers, x, y are rational. Let p_i, q_i be integers such that $0 < p_i < q_i$ and

$$x = \frac{p_1}{q_1} \quad , \quad y = \frac{p_2}{q_2} \ .$$

Thus we have derived a necessary condition for X to be n-ray partitional:

$$q_1 \text{ divides } \frac{n}{2}p_1 \quad \text{ and } \quad q_2 \text{ divides } \frac{n}{2}p_2 \ .$$

Conversely, if $X = (x, y)$ satisfies the above condition, then it is n-ray partitional since we can define a, b, c, d in terms of n, x, y as above.

To count the points that are 100-ray partitional, it suffices to count the ordered pairs of rationals $(\frac{p_1}{q_1}, \frac{p_2}{q_2})$ in the interior of R such that q_1 divides $\frac{100}{2}p_1$ and q_2 divides $\frac{100}{2}p_2$. This is just $\left\{(\frac{p_1}{50}, \frac{p_2}{50}) \ : \ 0 < p_1, p_2 < 50\right\}$ which has 49^2 points.

We need to subtract the points that are 100-ray partitional and 60-ray partitional; those are the ordered pairs of rationals $(\frac{p_1}{q_1}, \frac{p_2}{q_2})$ in the interior of R such that q_1 divides $GCD(\frac{100}{2}p_1, \frac{60}{2}p_1) = 10p_1$ and q_2 divides $GCD(\frac{100}{2}p_2, \frac{60}{2}p_2) = 10p_2$. This is just $\left\{(\frac{p_1}{10}, \frac{p_2}{10}) \ : \ 0 < p_1, p_2 < 10\right\}$ which has 9^2 points.

Hence the answer is $49^2 - 9^2 = \boxed{2320}$.

See Also

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