

2012 AMC 10A Problems/Problem 1

The following problem is from both the 2012 AMC 12A #2 and 2012 AMC 10A #1, so both problems redirect to this page.

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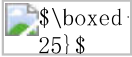
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Problem

Cagney can frost a cupcake every 20 seconds and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?

(A) 10 (B) 15 (C) 20 (D) 25 (E) 30

Solution 1

Cagney can frost one in **20** seconds, and Lacey can frost one in **30** seconds. Working together, they can frost one in $\frac{20 \cdot 30}{20 + 30} = \frac{600}{50} = 12$ seconds. In **300** seconds (**5** minutes), they can frost  cupcakes.

Solution 2

In **300** seconds (**5** minutes), Cagney will frost $\frac{300}{20} = 15$ cupcakes, and Lacey will frost $\frac{300}{30} = 10$ cupcakes. Therefore, working together they will frost $15 + 10 = \boxed{\text{(D) } 25}$ cupcakes.

Solution 3

Since Cagney frosts **3** cupcakes a minute, and Lacey frosts **2** cupcakes a minute, they together frost $3 + 2 = 5$ cupcakes a minute. Therefore, in **5** minutes, they frost $5 \times 5 = 25 \Rightarrow \boxed{\text{(D)}}$

See Also

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2012 AMC 10A Problems/Problem 2

Problem

A square with side length 8 is cut in half, creating two congruent rectangles. What are the dimensions of one of these rectangles?

- (A) 2 by 4 (B) 2 by 6 (C) 2 by 8 (D) 4 by 4 (E) 4 by 8

Solution

Cutting the square in half will bisect one pair of sides while the other side will remain unchanged. Thus, the new square is $\frac{8}{2} * 8$, or **(E) 4 by 8**.

See Also

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2012 AMC 10A Problems/Problem 3

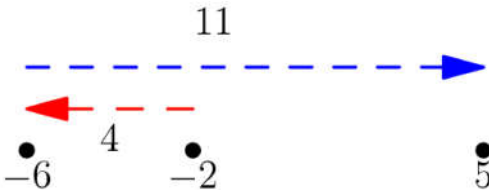
The following problem is from both the 2012 AMC 12A #1 and 2012 AMC 10A #3, so both problems redirect to this page.

Problem

A bug crawls along a number line, starting at -2 . It crawls to -6 , then turns around and crawls to 5 . How many units does the bug crawl altogether?

(A) 9 (B) 11 (C) 13 (D) 14 (E) 15

Solution



Crawling from -2 to -6 takes it a distance of 4 units. Crawling from -6 to 5 takes it a distance of 11 units. Add 4 and 11 to get **(E) 15**

See Also

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2012 AMC 10A Problems/Problem 4

Problem

Let $\angle ABC = 24^\circ$ and $\angle ABD = 20^\circ$. What is the smallest possible degree measure for angle CBD?
(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 12

Solution

$\angle ABD$ and $\angle ABC$ share ray AB . In order to minimize the value of $\angle CBD$, D should be located between A and C .
 $\angle ABC = \angle ABD + \angle CBD$, so $\angle CBD = 4$. The answer is **(C)** 4

See Also

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Category: Introductory Geometry Problems

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2012 AMC 10A Problems/Problem 5

Problem

Last year 100 adult cats, half of whom were female, were brought into the Smallville Animal Shelter. Half of the adult female cats were accompanied by a litter of kittens. The average number of kittens per litter was 4. What was the total number of cats and kittens received by the shelter last year?

- (A) 150
- (B) 200
- (C) 250
- (D) 300
- (E) 400

Solution

Half of the 100 adult cats are female, so there are $\frac{100}{2} = 50$ female cats. Half of those female adult cats have a litter of kittens, so there would be $\frac{50}{2} = 25$ litters. Since the average number of kittens per litter is 4, this implies that there are $25 \times 4 = 100$ kittens. So the total number of cats and kittens would be $100 + 100 =$

(B) 200

See Also

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2012 AMC 10A Problems/Problem 6

Problem

The product of two positive numbers is 9. The reciprocal of one of these numbers is 4 times the reciprocal of the other number. What is the sum of the two numbers?

- (A) $\frac{10}{3}$ (B) $\frac{20}{3}$ (C) 7 (D) $\frac{15}{2}$ (E) 8

Solution

Let the two numbers equal x and y . From the information given in the problem, two equations can be written:

$$xy = 9$$

$$\frac{1}{x} = 4 \left(\frac{1}{y} \right)$$

Therefore, $4x = y$

Replacing y with $4x$ in the equation,

$$4x^2 = 9$$

So $x = \frac{3}{2}$ and y would then be $4 \times \frac{3}{2} = 6$

The sum would be $\frac{3}{2} + 6 = \boxed{\text{(D)} \frac{15}{2}}$

See Also

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2012 AMC 10A Problems/Problem 7

The following problem is from both the 2012 AMC 12A #4 and 2012 AMC 10A #7, so both problems redirect to this page.

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Problem

In a bag of marbles, $\frac{3}{5}$ of the marbles are blue and the rest are red. If the number of red marbles is doubled and the number of blue marbles stays the same, what fraction of the marbles will be red?

- (A) $\frac{2}{5}$
- (B) $\frac{3}{7}$
- (C) $\frac{4}{7}$
- (D) $\frac{3}{5}$
- (E) $\frac{4}{5}$

Solution 1

Assume that there are 5 total marbles in the bag. The actual number does not matter, since all we care about is the ratios, and the only operation performed on the marbles in the bag is doubling.

There are 3 blue marbles in the bag and 2 red marbles. If you double the amount of red marbles, there will still be 3 blue marbles but now there will be 4 red marbles. Thus, the answer is

(C) $\frac{4}{7}$

.

Solution 2

Let us say that there are x marbles in the bag. Therefore, $\frac{3x}{5x}$ are blue, and $\frac{2x}{5x}$ are red. When the red marbles are doubled, we now have $\frac{2 * 2x}{5x + 2x} = \frac{4x}{7x} = \frac{4}{7} \Rightarrow$

(C)

See Also

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2012 AMC 10A Problems/Problem 8

The following problem is from both the 2012 AMC 12A #6 and 2012 AMC 10A #8, so both problems redirect to this page.

Problem

The sums of three whole numbers taken in pairs are 12, 17, and 19. What is the middle number?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Solution

Let the three numbers be equal to a , b , and c . We can now write three equations:

$$a + b = 12$$

$$b + c = 17$$

$$a + c = 19$$

Adding these equations together, we get that

$$2(a + b + c) = 48 \text{ and}$$

$$a + b + c = 24$$

Substituting the original equations into this one, we find

$$c + 12 = 24$$

$$a + 17 = 24$$

$$b + 19 = 24$$

Therefore, our numbers are 12, 7, and 5. The middle number is (D) 7

See Also

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2012 AMC 10A Problems/Problem 9

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Problem

A pair of six-sided dice are labeled so that one die has only even numbers (two each of 2, 4, and 6), and the other die has only odd numbers (two of each 1, 3, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on the tops of the two dice is 7?

- (A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

Solution

To solve this, we need to find the number of ways that we can roll a sum of 7 divided by the total possible rolls.

The total number of combinations when rolling two dice is $6 * 6 = 36$.

There are three ways that a sum of 7 can be rolled, $2 + 5$, $4 + 3$, and $6 + 1$. There are two 2's on one die and two 5's on the other, so there are a total of 4 ways to roll the combination of 2 and 5. There are two 4's on one die and two 3's on the other, so there are a total of 4 ways to roll the combination of 4 and 3. There are two 6's on one die and two 1's on the other, so there are a total of 4 ways to roll the combination of 6 and 1. Add $4 + 4 + 4 = 12$.

Thus, our probability is $\frac{12}{36} = \frac{1}{3}$. The answer is (D) $\frac{1}{3}$.

Solution 2

Assume we roll the die with only evens first. For whatever value rolled, there are exactly 2 faces on the odd die that makes the sum 7. The odd die has 6 faces, so our probability is (D) $\frac{1}{3}$.

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2012 AMC 10A Problems/Problem 10

The following problem is from both the 2012 AMC 12A #7 and 2012 AMC 10A #10, so both problems redirect to this page.

Problem

Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

(A) 5 (B) 6 (C) 8 (D) 10 (E) 12

Solution

If we let a be the smallest sector angle and r be the difference between consecutive sector angles, then we have the angles $a, a + r, a + 2r, \dots, a + 11r$. Use the formula for the sum of an arithmetic sequence and set it equal to 360, the number of degrees in a circle.

$$\begin{aligned}\frac{a + a + 11r}{2} \cdot 12 &= 360 \\ 2a + 11r &= 60 \\ a &= \frac{60 - 11r}{2}\end{aligned}$$

All sector angles are integers so r must be a multiple of 2. Plug in even integers for r starting from 2 to minimize a . We find this value to be 4 and the minimum value of a to be $\frac{60 - 11(4)}{2} = \boxed{\text{(C) } 8}$

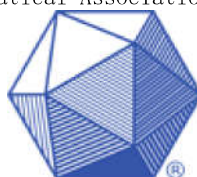
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Category: Introductory Geometry Problems

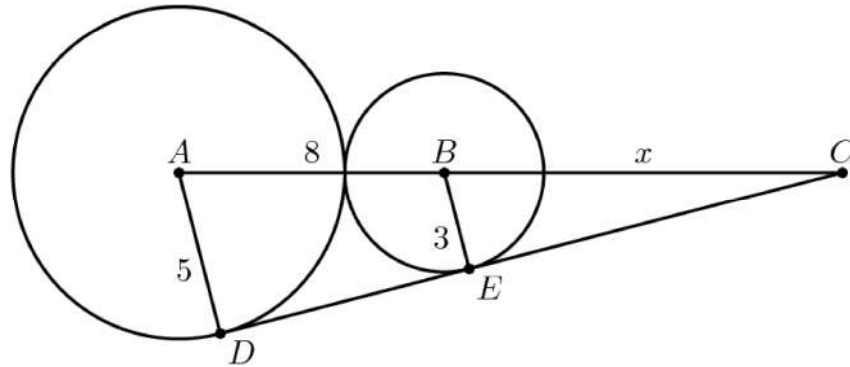
2012 AMC 10A Problems/Problem 11

Problem

Externally tangent circles with centers at points A and B have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray AB at point C . What is BC ?

- (A) 4 (B) 4.8 (C) 10.2 (D) 12 (E) 14.4

Solution



Let D and E be the points of tangency on circles A and B with line CD . $AB = 8$. Also, let $BC = x$. As $\angle ADC$ and $\angle BEC$ are right angles (a radius is perpendicular to a tangent line at the point of tangency) and both triangles share $\angle ACD$, $\triangle ADC \sim \triangle BEC$. From this we can get a proportion.

$$\frac{BC}{AC} = \frac{BE}{AD} \rightarrow \frac{x}{x+8} = \frac{3}{5} \rightarrow 5x = 3x + 24 \rightarrow x = \boxed{\text{(D) } 12}$$

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2012 AMC 10A Problems/Problem 12

The following problem is from both the 2012 AMC 12A #9 and 2012 AMC 10A #12, so both problems redirect to this page.

Problem

A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not 100 (such as 2012). The 200th anniversary of the birth of novelist Charles Dickens was celebrated on February 7, 2012, a Tuesday. On what day of the week was Dickens born?

- (A) Friday (B) Saturday (C) Sunday (D) Monday (E) Tuesday

Solution

Each year we go back is one day back, because $365 \equiv 1 \pmod{7}$. Each leap year we go back is two days back, since $366 \equiv 2 \pmod{7}$. A leap year is usually every four years, so 200 years would have $\frac{200}{4} = 50$ leap years, but the problem points out that 1900 does not count as a leap year.

This would mean a total of 151 regular years and 49 leap years, so $1(151) + 2(49) = 249$ days back. Since $249 \equiv 4 \pmod{7}$, four days back from Tuesday would be **(A) Friday**.

See Also

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2012 AMC 10A Problems/Problem 13

The following problem is from both the 2012 AMC 12A #8 and 2012 AMC 10A #13, so both problems redirect to this page.

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Problem

An iterative average of the numbers 1, 2, 3, 4, and 5 is computed the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

(A) $\frac{31}{16}$ (B) 2 (C) $\frac{17}{8}$ (D) 3 (E) $\frac{65}{16}$

Solution 1

The minimum and maximum can be achieved with the orders 5, 4, 3, 2, 1 and 1, 2, 3, 4, 5.

$$5, 4, 3, 2, 1 \Rightarrow \frac{9}{2}, 3, 2, 1 \Rightarrow \frac{15}{4}, 2, 1 \Rightarrow \frac{23}{8}, 1 \Rightarrow \frac{31}{16}$$

$$1, 2, 3, 4, 5 \Rightarrow \frac{3}{2}, 3, 4, 5 \Rightarrow \frac{9}{4}, 4, 5 \Rightarrow \frac{25}{8}, 5 \Rightarrow \frac{65}{16}$$

The difference between the two is $\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \boxed{\text{(C)} \frac{17}{8}}$.

Solution 2

The iterative average of any 5 integers a, b, c, d, e can be thought of as:

$$((((\frac{a+b}{2}) + c)/2 + d)/2 + e)/2$$

Expanding this, we see that this fraction is equal to:

$$\frac{a + b + 2c + 4d + 8e}{16}$$

Plugging in 1, 2, 3, 4, 5 for a, b, c, d, e , we see that in order to maximize the fraction,

$$a = 1, b = 2, c = 3, d = 4, e = 5,$$

and in order to minimize the fraction,

$$a = 5, b = 4, c = 3, d = 2, e = 1.$$

After plugging in these values and finding the positive difference of the two fractions, we arrive with $\frac{34}{16} \Rightarrow \frac{17}{8}$, which is our answer of $\boxed{\text{(C)}}$

2012 AMC 10A Problems/Problem 14

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Problem

Chubby makes nonstandard checkerboards that have **31** squares on each side. The checkerboards have a black square in every corner and alternate red and black squares along every row and column. How many black squares are there on such a checkerboard?

(A) 480 (B) 481 (C) 482 (D) 483 (E) 484

Solution 1

There are 15 rows with 15 black tiles, and 16 rows with 16 black tiles, so the answer is

$$15^2 + 16^2 = 225 + 256 = \boxed{\text{(B)} 481}$$

Solution 2

We build the **31** $\times$ **31** checkerboard starting with a board of **30** $\times$ **30** that is exactly half black. There are **15** $\cdot$ **30** black tiles in this region.

Add to this **30** $\times$ **30** checkerboard a **1** $\times$ **30** strip on the bottom that has **15** black tiles.

Add to this **31** $\times$ **30** checkerboard a **31** $\times$ **1** strip on the right that has **15** + **1** black tiles.

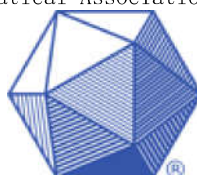
In total, there are **15** $\cdot$ **30** + **15** + **15** + **1** = **481** tiles, giving an answer of **(B) 481**

See Also

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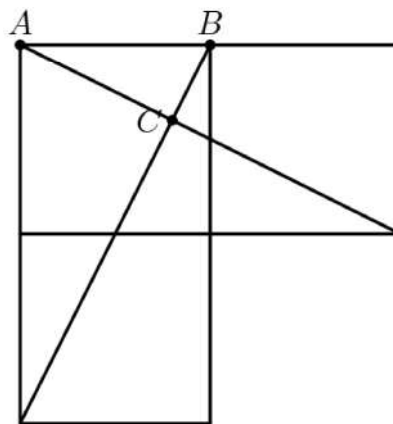
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- 6 See Also

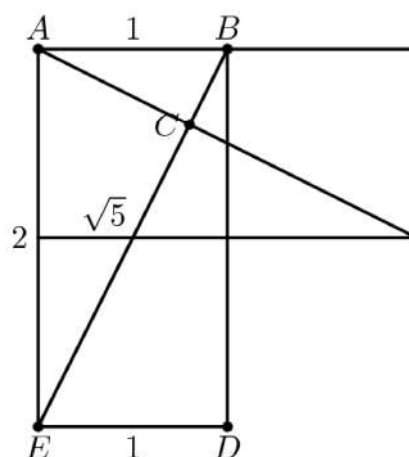
Problem

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of $\triangle ABC$?



- (A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{2}{9}$ (D) $\frac{1}{3}$ (E) $\frac{\sqrt{2}}{4}$

Solution 1



AC intersects BE at a right angle, so $\triangle ABC \sim \triangle BED$. The hypotenuse of right triangle BED is $\sqrt{1^2 + 2^2} = \sqrt{5}$.

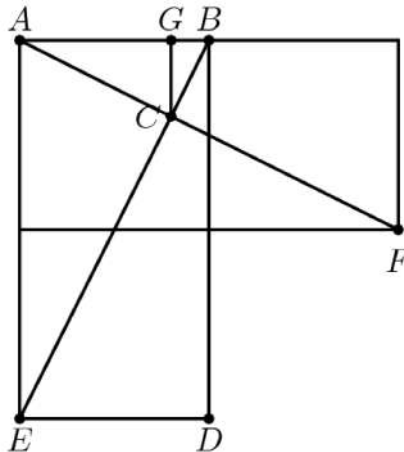
$$\frac{AC}{BC} = \frac{BD}{ED} \Rightarrow \frac{AC}{BC} = \frac{2}{1} \Rightarrow AC = 2BC$$

$$\frac{AC}{AB} = \frac{BD}{BE} \Rightarrow \frac{AC}{1} = \frac{2}{\sqrt{5}} \Rightarrow AC = \frac{2}{\sqrt{5}}$$

Since $AC=2BC$, $BC = \frac{1}{\sqrt{5}}$. $\triangle ABC$ is a right triangle so the area is just

$$\frac{1}{2} \cdot AC \cdot BC = \frac{1}{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \boxed{\text{(B)} \frac{1}{5}}$$

Solution 2



Let **E** be the origin. Then, $D = (1, 0)$ $A = (0, 2)$ $B = (1, 2)$ $F = (2, 1)$

EB can be represented by the line $y = 2x$ Also, AF can be represented by the line $y = -\frac{1}{2}x + 2$

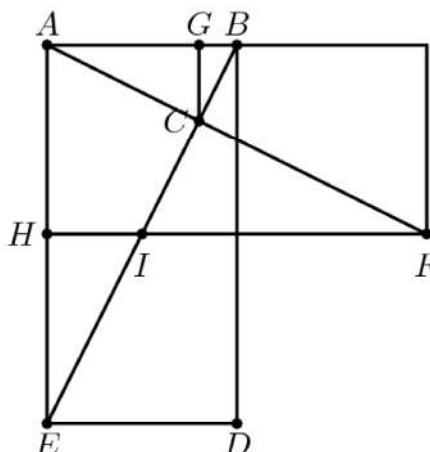
Subtracting the second equation from the first gives us $\frac{5}{2}x - 2 = 0$. Thus, $x = \frac{4}{5}$. Plugging this into the first equation gives us $y = \frac{8}{5}$.

Since $C(0.8, 1.6)$, G is $(0.8, 2)$.

$AB = 1$ and $CG = 0.4$.

Thus, $[ABC] = \frac{1}{2} \cdot AB \cdot CG = \frac{1}{2} \cdot 1 \cdot 0.4 = 0.2 = \frac{1}{5}$. The answer is **(B)**.

Solution 3



Triangle EAB is similar to triangle EHI ; line $HI = 1/2$

Triangle ACB is similar to triangle FCI and the ratio of line AB to line $IF = 1 : \frac{3}{2} = 2 : 3$.

Based on similarity the length of the height of GC is thus $\frac{2}{5} \cdot 1 = \frac{2}{5}$.

Thus, $[ABC] = \frac{1}{2} \cdot AB \cdot CG = \frac{1}{2} \cdot 1 \cdot \frac{2}{5} = \frac{1}{5}$. The answer is (B)

Solution 4

Let L be the point where the diagonal and the end of the unit square meet, on the right side of the diagram. Let K be the top right corner of the top right unit square, where segment ABL is 2 units in length.

Because of the Pythagorean Theorem, since $AC = 2$ and $LK = 1$, the diagonal of triangle ALK is $\sqrt{5}$.

Triangle ALK is clearly a similar triangle to triangle ABC . Segment AB is the hypotenuse of triangle ABC . So, we can write down:

$$AK/AB = LK/BC$$

, which is equal to:

$$\frac{\sqrt{5}}{1} = \frac{1}{BC}$$

Solving this equation yields:

$$BC = \frac{1}{\sqrt{5}}$$

By Pythagorean theorem, we can now find segment AC

$$\left(\frac{1}{\sqrt{5}}\right)^2 + AC^2 = 1$$

Solving this yields:

$$AC^2 = \frac{4}{5}$$

$$\text{, so } AC = \frac{2}{\sqrt{5}}$$

So then we can use

$$A = \frac{1}{2} * b * a.$$

So

$$A = \frac{1}{2} * \frac{1}{\sqrt{5}} * \frac{2}{\sqrt{5}}$$

$$= \frac{1}{5} \text{ or } \boxed{\text{B}}$$

2012 AMC 10A Problems/Problem 16

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4
- 6 See Also

Problem

Three runners start running simultaneously from the same point on a 500-meter circular track. They each run clockwise around the course maintaining constant speeds of 4.4, 4.8, and 5.0 meters per second. The runners stop once they are all together again somewhere on the circular course. How many seconds do the runners run?

(A) 1,000 (B) 1,250 (C) 2,500 (D) 5,000 (E) 10,000

Solution 1

First consider the first two runners. The faster runner will lap the slower runner exactly once, or run 500 meters farther. Let x be the time these runners run in seconds.

$$4.8x - 4.4x = 500 \Rightarrow x = 1250$$

Because $4.4(1250) = 5500$ is a multiple of 500, it turns out they just meet back at the start line.

Now we must find a time that is a multiple of 1250 and results in the 5.0 m/s runner to end up on the start line. Every 1250 seconds, that fastest runner goes $5.0(1250) = 6250$ meters. In $2(1250) = 2500$ seconds, he goes $5.0(2500) = 12500$ meters. Therefore the runners run (C) 2,500 seconds.

Solution 2

Working backwards from the answers starting with the smallest answer, if they had run 1000 seconds, they would have run 4400, 4800, 5000 meters, respectively. The first two runners have a difference of 400 meters, which is not a multiple of 500 (one lap), so they are not in the same place.

If they had run 1250 seconds, the runners would have run 5500, 6000, 6250 meters, respectively. The last two runners have a difference of 250 meters, which is not a multiple of 500.

If they had run 2500 seconds, the runners would have run 11000, 12000, 12500 meters, respectively. The distance separating each pair of runners is a multiple of 500, so the answer is (C) 2,500 seconds.

Solution 3

Let t be the time run in seconds, then the difference in meters run between the three runners is $0.2t, 0.4t, 0.6t$. For them to be at the same location all of them need to be multiples of 500. It is now easy to see that

$$0.2t = 500, 0.4t = 1000, 0.6t = 1500, \text{ so } t = \span style="border: 1px solid black; padding: 2px;">(C) 2,500.$$

Solution 4

After t seconds, respectively the runners would've ran $4.4t, 4.8t$, and $5t$ meters. Their current positions on the track are these values $(\text{mod } 500)$. We're trying to find the value of t such that

$$4.4t \equiv 4.8t \equiv 5t \pmod{500}$$

Subtracting $4.4t$ on all sides, we get

$$0 \equiv 0.4t \equiv 0.6t \pmod{500}$$

Now, we must find a value for t such that both $0.6t$ and $0.4t$ are simultaneously multiples of 500.

Plugging in 500 for $0.4t$ we get $t = 1250$, but this does not work for $0.6t$ (750 isn't a multiple of 500).
Plugging in $0.4t = 1000$, we get $t = 2500$, and this does work for $0.6t$.

Therefore, $t = 2500$ and the answer is **(C) 2500**.

— @adihaya (talk) 15:21, 19 February 2016 (EST)

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 10A Problems/Problem 17

Contents

- 1 Problem
- 2 Solution 1 (Meta)
- 3 Solution 2
- 4 Solution 3
- 5 See Also

Problem

Let a and b be relatively prime integers with $a > b > 0$ and $\frac{a^3 - b^3}{(a - b)^3} = \frac{73}{3}$. What is $a - b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution 1 (Meta)

Since a and b are both integers, $a^3 - b^3$ and $(a - b)^3$ are both integers as well. Then, for the given fraction to simplify to $\frac{73}{3}$, the denominator $(a - b)^3$ must be a multiple of 3. Thus, $a - b$ is a multiple of 3. Looking at the answer choices, the only multiple of 3 is **(C) 3**.

Solution 2

Using difference of cubes in the numerator and cancelling out one $(a - b)$ in the numerator and denominator gives $\frac{a^2 + ab + b^2}{a^2 - 2ab + b^2} = \frac{73}{3}$.

Set $x = a^2 + b^2$, and $y = ab$. Then $\frac{x + y}{x - 2y} = \frac{73}{3}$. Cross multiplying gives $3x + 3y = 73x - 146y$,

and simplifying gives $\frac{x}{y} = \frac{149}{70}$. Since 149 and 70 are relatively prime, we let $x = 149$ and $y = 70$, giving $a^2 + b^2 = 149$ and $ab = 70$. Since $a > b > 0$, the only solution is $(a, b) = (10, 7)$, which can be seen upon squaring and summing the various factor pairs of 70.

Thus, $a - b = \mathbf{(C) 3}$.

Remarks:

An alternate method of solving the system of equations involves solving the second equation for a , by plugging it into the first equation, and solving the resulting quartic equation with a substitution of $u = b^2$. The four solutions correspond to $(\pm 10, \pm 7), (\pm 7, \pm 10)$.

Also, we can solve for $a - b$ directly instead of solving for a and b :
 $a^2 - 2ab + b^2 = 149 - 2(70) = 9 \implies a - b = 3$.

Note that if you double x and double y , you will get different (but not relatively prime) values for a and b that satisfy the original equation.

Solution 3

The first step is the same as above which gives $\frac{a^2 + ab + b^2}{a^2 - 2ab + b^2} = \frac{73}{3}$.

Then we can subtract $3ab$ and then add $3ab$ to get $\frac{a^2 - 2ab + b^2 + 3ab}{a^2 - 2ab + b^2} = \frac{73}{3}$, which gives

$$1 + \frac{3ab}{(a-b)^2} = \frac{73}{3} \implies \frac{3ab}{(a-b)^2} = \frac{70}{3}.$$

Cross multiply $9ab = 70(a-b)^2$. Since $a > b$, take the square root. $a-b = 3\sqrt{\frac{ab}{70}}$. Since a and b are integers and relatively prime, $\sqrt{\frac{ab}{70}}$ is an integer. ab is a multiple of 70 , so $a-b$ is a multiple of 3 . Therefore $a = 10$ and $b = 7$ is a solution. So

$a - b = \boxed{(C) \ 3}$

See Also

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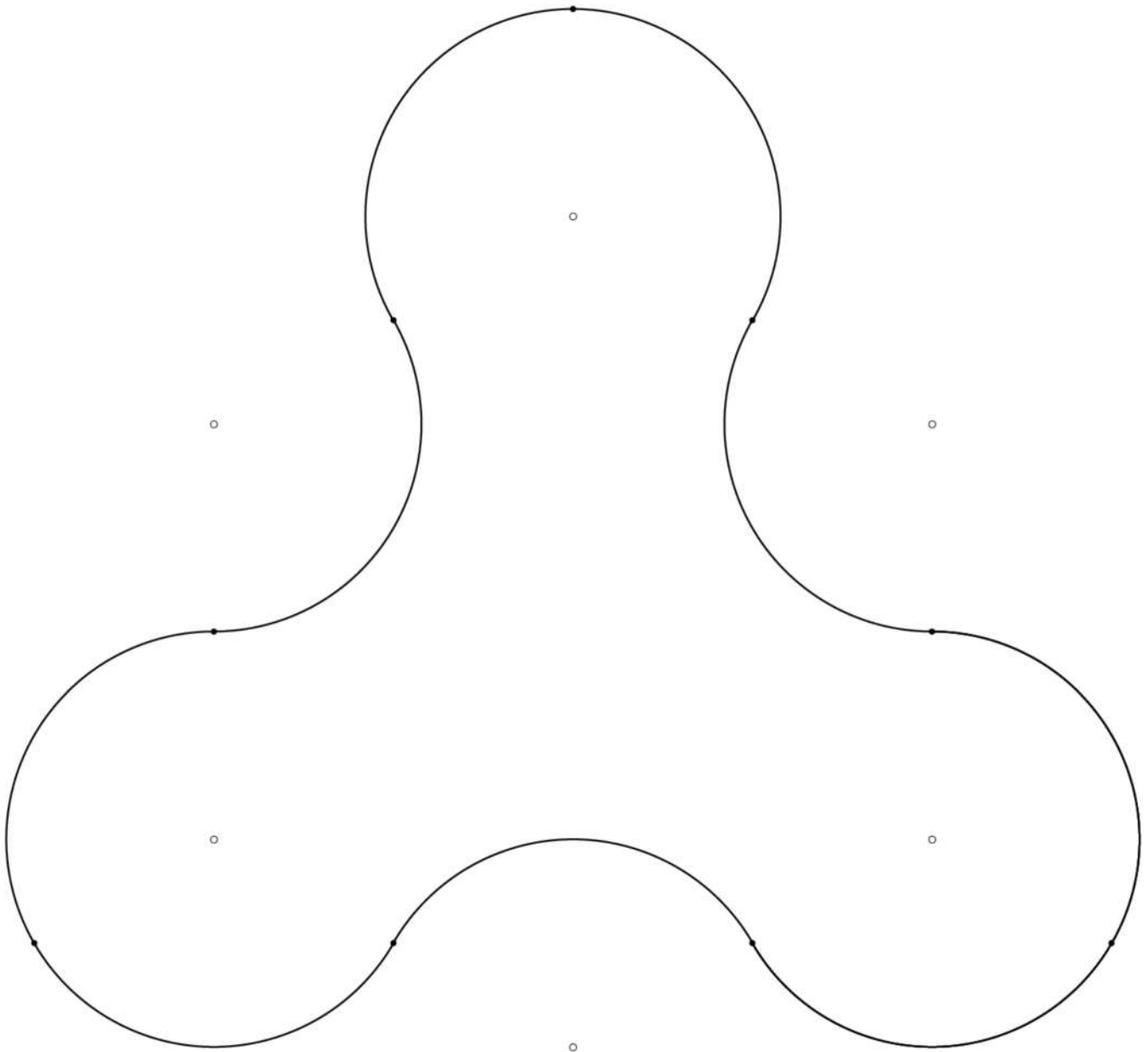
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2012 AMC 10A Problems/Problem 18

The following problem is from both the 2012 AMC 12A #14 and 2012 AMC 10A #18, so both problems redirect to this page.

Problem 18

The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



- (A) $2\pi + 6$ (B) $2\pi + 4\sqrt{3}$ (C) $3\pi + 4$ (D) $2\pi + 3\sqrt{3} + 2$ (E) $\pi + 6\sqrt{3}$

Solution

Draw the hexagon between the centers of the circles, and compute its area $(6)(0.5)(2\sqrt{3}) = 6\sqrt{3}$. Then add the areas of the three sectors outside the hexagon (2π) and subtract the areas of the three sectors inside the hexagon but outside the figure (π) to get the area enclosed in the curved figure $(\pi + 6\sqrt{3})$, which is **(E)** $\pi + 6\sqrt{3}$.

2012 AMC 10A Problems/Problem 19

The following problem is from both the 2012 AMC 12A #13 and 2012 AMC 10A #19, so both problems redirect to this page.

Problem 19

Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

(A) 30 (B) 36 (C) 42 (D) 48 (E) 60

Solution

Let Paula work at a rate of p , the two helpers work at a combined rate of h , and the time it takes to eat lunch be L , where p and h are in house/hours and L is in hours. Then the labor on Monday, Tuesday, and Wednesday can be represented by the three following equations:

$$(8 - L)(p + h) = .50$$

$$(6.2 - L)h = .24$$

$$(11.2 - L)p = .26$$

With three equations and three variables, we need to find the value of L . Adding the second and third equations together gives us $6.2h + 11.2p - L(p + h) = .50$. Subtracting the first equation from this new one gives us $-1.8h + 3.2p = 0$, so we get $h = \frac{16}{9}p$. Plugging into the second equation:

$$(6.2 - L)\frac{16}{9}p = .24$$

$$(6.2 - L)p = \frac{27}{200}$$

We can then subtract this from the third equation:

$$5p = .26 - \frac{27}{200}$$

$$p = \frac{1}{40}$$

Plugging p into our third equation gives:

$$L = \frac{4}{5}$$

Converting L from hours to minutes gives us $L = 48$ minutes, which is (D) 48.

See Also

2012 AMC 10A Problems/Problem 20

The following problem is from both the 2012 AMC 12A #15 and 2012 AMC 10A #20, so both problems redirect to this page.

Problem

A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black?

- (A) $\frac{49}{512}$ (B) $\frac{7}{64}$ (C) $\frac{121}{1024}$ (D) $\frac{81}{512}$ (E) $\frac{9}{32}$

Solution

First, there is only one way for the middle square to be black because it is not affected by the rotation. Then we can consider the corners and edges separately. Let's first just consider the number of ways we can color the corners. There is 1 case with all black squares. There are four cases with one white square and all 4 work. There are six cases with two white squares, but only the 2 with the white squares diagonal from each other work. There are no cases with three white squares or four white squares. Then the total number of ways to color the corners is $1 + 4 + 2 = 7$. In essence, the edges work the same way, so there are also 7 ways to color them. The number of ways to fit the conditions over the number of ways to color the squares is

$$\frac{7 \times 7}{2^9} = \boxed{\text{(A)} \frac{49}{512}}$$

See Also

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2012 AMC 10A Problems/Problem 21

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- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See Also

Problem

Let points $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 2, 0)$, and $D = (0, 0, 3)$. Points E , F , G , and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of $EFGH$?

- (A) $\sqrt{2}$ (B) $\frac{2\sqrt{5}}{3}$ (C) $\frac{3\sqrt{5}}{4}$ (D) $\sqrt{3}$ (E) $\frac{2\sqrt{7}}{3}$

Solution 1

Consider a tetrahedron with vertices at A, B, C, D on the xyz -plane. The length of EF is just one-half of AD because it is the midsegment of $\triangle ABD$. The same concept applies to the other side lengths. $AD = 3$ and $BC = \sqrt{1^2 + 2^2} = \sqrt{5}$. Then $EF = HG = \frac{3}{2}$ and $EH = FG = \frac{\sqrt{5}}{2}$. The line segments lie on perpendicular planes so quadrilateral $EFGH$ is a rectangle. The area is

$$EF \cdot FG = \frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \boxed{\text{(C)} \frac{3\sqrt{5}}{4}}$$

Solution 2

Computing the points of $EFGH$ gives $E(0.5, 0, 1.5)$, $F(0.5, 0, 0)$, $G(0, 1, 0)$, $H(0, 1, 1.5)$. The vector EF is $(0, 0, -1.5)$, while the vector HG is also $(0, 0, -1.5)$, meaning the two sides EF and GH are parallel. Similarly, the vector FG is $(-0.5, 1, 0)$, while the vector EH is also $(-0.5, 1, 0)$. Again, these are equal in both magnitude and direction, so FG and EH are parallel. Thus, figure $EFGH$ is a parallelogram.

Computation of vectors EF and HG is sufficient evidence that the figure is a parallelogram, since the vectors are not only point in the same direction, but are of the same magnitude, but the other vector FG is needed to find the angle between the sides.

Taking the dot product of vector EF and vector FG gives $0 \cdot -0.5 + 0 \cdot 1 + -1.5 \cdot 0 = 0$, which means the two vectors are perpendicular. (Alternately, as above, note that vector EF goes directly down on the z -axis, while vector FG has no z -component and lie completely in the xy plane.) Thus, the figure is a parallelogram with a right angle, which makes it a rectangle. With the distance formula in three dimensions, we find that $EF = \frac{3}{2}$

and $FG = \frac{\sqrt{5}}{2}$, giving an area of $\frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \boxed{\text{(C)} \frac{3\sqrt{5}}{4}}$

See Also

2012 AMC 10A Problems/Problem 22

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- 1 Problem
- 2 Solution 1
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Problem

The sum of the first m positive odd integers is 212 more than the sum of the first n positive even integers. What is the sum of all possible values of n ?

(A) 255 (B) 256 (C) 257 (D) 258 (E) 259

Solution 1

The sum of the first m odd integers is given by m^2 . The sum of the first n even integers is given by $n(n+1)$.

Thus, $m^2 = n^2 + n + 212$. Since we want to solve for n , rearrange as a quadratic equation:
 $n^2 + n + (212 - m^2) = 0$.

Use the quadratic formula: $n = \frac{-1 + \sqrt{1 - 4(212 - m^2)}}{2}$. n is clearly an integer, so

$1 - 4(212 - m^2) = 4m^2 - 847$ must be not only a perfect square, but also an odd perfect square. This is because the entire expression must be an integer, and for the numerator to be even (divisible by 2), $4m^2 - 847$ must be odd.

Let $x = \sqrt{4m^2 - 847}$. (Note that this means that $n = \frac{-1 + x}{2}$.) This can be rewritten as
 $x^2 = 4m^2 - 847$, which can then be rewritten to $4m^2 - x^2 = 847$. Factor the left side by using the difference of squares. $(2m + x)(2m - x) = 847 = 7 \cdot 11^2$.

Our goal is to find possible values for x , then use the equation above to find n . The difference between the factors is $(2m + x) - (2m - x) = 2m + x - 2m + x = 2x$. We have three pairs of factors, $847 \cdot 1$, $7 \cdot 121$, and $11 \cdot 77$. The differences between these factors are 846, 114, and 66 – those are all possible values for $2x$. Thus the possibilities for x are 423, 57, and 33.

Now plug in these values into the equation $n = \frac{-1 + x}{2}$. n can equal 211, 28, or 16. Add
 $211 + 28 + 16 = 255$. The answer is (A) 255.

Solution 2

As above, start off by noting that the sum of the first m odd integers $= m^2$ and the sum of the first n even integers $= n(n+1)$. Clearly $m > n$, so let $m = n + a$, where a is some positive integer. We have:

$(n + a)^2 = n(n + 1) + 212$. Expanding, grouping like terms and factoring, we get:
 $n = (212 - a^2)/(2a - 1)$.

We know that n and a are both positive integers, so we need only check values of a from 1 to 14 ($14^2 = 196 < 212 < 15^2 = 225$). Plugging in, the only values of a that give integral solutions are 1, 4, and 6. These gives n values of 211, 28, and 16, respectively. $211 + 28 + 16 = 255$. Hence, the answer is (A) 255.

Solution 3

Using the closed forms for the sums, we get $m^2 = n(n + 1) + 212$, or $m^2 = n^2 + n + 212$. We would like to factor this equation, but the current expressions don't allow for this. So we multiply both sides by 4 to let us complete the square. Our equation is now $4m^2 = 4n^2 + 4n + 848$. Complete the square on the right hand side: $4m^2 = (4n^2 + 4n + 1) + 848 - 1 = (2n + 1)^2 + 847$. Move over the $(2n + 1)^2$ and factor to get $(2m - 2n - 1)(2m + 2n + 1) = 847 = 7 * 11 * 11$. The second factor is clearly greater than the first, and the only possible factor pairs are 1 and 847, 7 and 121, 11 and 77. In each of these cases, solve for m and n and we find the solutions $(m, n) = (212, 211), (32, 28), (22, 16)$. The sum of all possible values of n is $211 + 28 + 16 = 255$.

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 10A Problems/Problem 23

The following problem is from both the 2012 AMC 12A #19 and 2012 AMC 10A #23, so both problems redirect to this page.

Problem

Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

- (A) 60
- (B) 170
- (C) 290
- (D) 320
- (E) 660

Solution

Note that if n is the number of friends each person has, then n can be any integer from 1 to 4 , inclusive.

Also note that the cases of $n = 1$ and $n = 4$ are the same, since a map showing a solution for $n = 1$ can correspond one-to-one with a map of a solution for $n = 4$ by simply making every pair of friends non-friends and vice versa. The same can be said of configurations with $n = 2$ when compared to configurations of $n = 3$. Thus, we have two cases to examine, $n = 1$ and $n = 2$, and we count each of these combinations twice.

For $n = 1$, if everyone has exactly one friend, that means there must be 3 pairs of friends, with no other interconnections. The first person has 5 choices for a friend. There are 4 people left. The next person has 3 choices for a friend. There are two people left, and these remaining two must be friends. Thus, there are 15 configurations with $n = 1$.

For $n = 2$, there are two possibilities. The group of 6 can be split into two groups of 3 , with each group creating a friendship triangle. The first person has $\binom{5}{2} = 10$ ways to pick two friends from the other five, while the other three are forced together. Thus, there are 10 triangular configurations.

However, the group can also form a friendship hexagon, with each person sitting on a vertex, and each side representing the two friends that person has. The first person may be seated anywhere on the hexagon Without loss of generality. This person has $\binom{5}{2} = 10$ choices for the two friends on the adjoining vertices. Each of the three remaining people can be seated "across" from one of the original three people, forming a different configuration. Thus, there are $10 \cdot 3! = 60$ hexagonal configurations, and in total 70 configurations for $n = 2$.

As stated before, $n = 3$ has 70 configurations, and $n = 4$ has 15 configurations. This gives a total of $(70 + 15) \cdot 2 = 170$ configurations, which is option **(B) 170**.

See Also

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2012 AMC 10A Problems/Problem 24

The following problem is from both the 2012 AMC 12A #21 and 2012 AMC 10A #24, so both problems redirect to this page.

Problem

Let a , b , and c be positive integers with $a \geq b \geq c$ such that $a^2 - b^2 - c^2 + ab = 2011$ and $a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997$.

What is a ?

- (A) 249 (B) 250 (C) 251 (D) 252 (E) 253

Solution

Add the two equations.

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc = 14.$$

Now, this can be rearranged:

$$(a^2 - 2ab + b^2) + (a^2 - 2ac + c^2) + (b^2 - 2bc + c^2) = 14$$

and factored:

$$(a - b)^2 + (a - c)^2 + (b - c)^2 = 14$$

a , b , and c are all integers, so the three terms on the left side of the equation must all be perfect squares. Recognize that $14 = 9 + 4 + 1$.

$(a - c)^2 = 9 \rightarrow a - c = 3$, since $a - c$ is the biggest difference. It is impossible to determine by inspection whether $a - b = 2$ or 1 , or whether $b - c = 1$ or 2 .

We want to solve for a , so take the two cases and solve them each for an expression in terms of a . Our two cases are $(a, b, c) = (a, a - 1, a - 3)$ or $(a, a - 2, a - 3)$. Plug these values into one of the original equations to see if we can get an integer for a .

$a^2 - (a - 1)^2 - (a - 3)^2 + a(a - 1) = 2011$, after some algebra, simplifies to $7a = 2021$. 2021 is not divisible by 7, so a is not an integer.

The other case gives $a^2 - (a - 2)^2 - (a - 3)^2 + a(a - 2) = 2011$, which simplifies to $8a = 2024$.

Thus, $a = 253$ and the answer is **(E) 253**.

See Also

2012 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 10A Problems/Problem 25

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Problem

Real numbers x , y , and z are chosen independently and at random from the interval $[0, n]$ for some positive integer n . The probability that no two of x , y , and z are within 1 unit of each other is greater than $\frac{1}{2}$. What is the smallest possible value of n ?

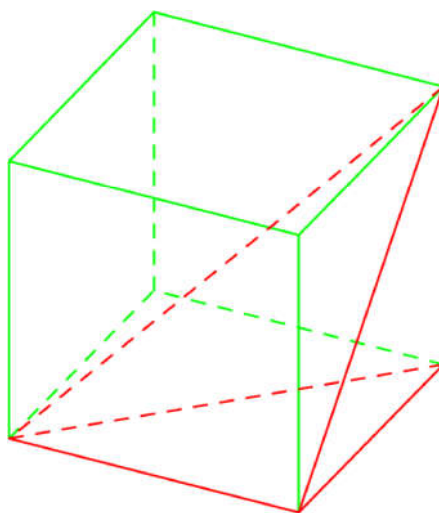
(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Solution I

Since x, y, z are all reals located in $[0, n]$, the number of choices for each one is infinite.

Without loss of generality, assume that $n \geq x \geq y \geq z \geq 0$. Then the set of points (x, y, z) is a tetrahedron, or a triangular pyramid. The point (x, y, z) distributes uniformly in this region. If this is not easy to understand, read Solution II.

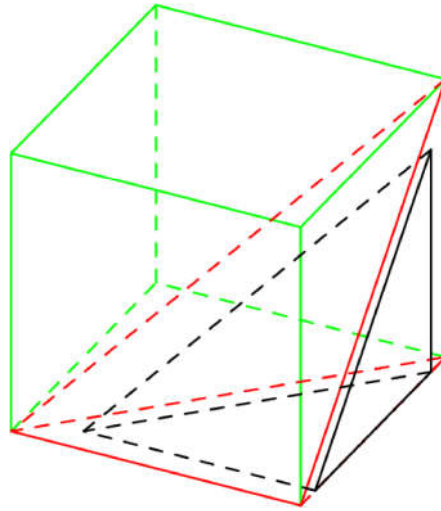
The altitude of the tetrahedron is n and the base is an isosceles right triangle with a leg length n . The volume is $V_1 = \frac{n^3}{6}$, as shown in the first figure in red.



Now we will find the region with points satisfying $|x - y| \geq 1$, $|y - z| \geq 1$, $|z - x| \geq 1$.

Since $n \geq x \geq y \geq z \geq 0$, we have $x - y \geq 1$, $y - z \geq 1$.

The region of points (x, y, z) satisfying the condition is shown in the second figure in black. It is a tetrahedron, too.



The volume of this region is $V_2 = \frac{(n-2)^3}{6}$.

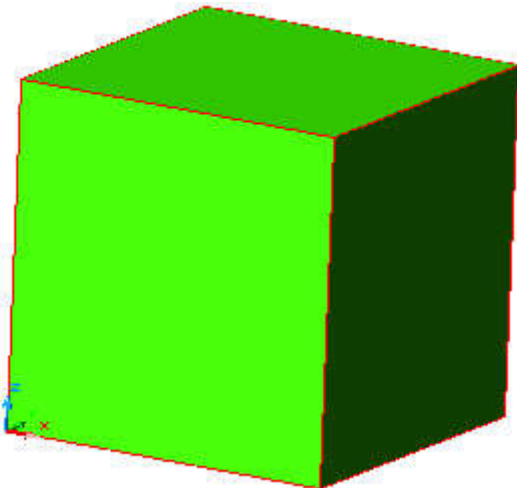
So the probability is $p = \frac{V_2}{V_1} = \frac{(n-2)^3}{n^3}$.

Substituting n with the values in the choices, we find that when $n = 10$, $p = \frac{512}{1000} > \frac{1}{2}$, when $n = 9$, $p = \frac{343}{729} < \frac{1}{2}$. So $n \geq 10$.

So the answer is (D) 10.

Solution II

Because x , y , and z are chosen independently and at random from the interval $[0, n]$, which means that x , y , and z distributes uniformly and independently in the interval $[0, n]$. So the point (x, y, z) distributes uniformly in the cubic $0 \leq x, y, z \leq n$, as shown in the figure below. The volume of this cubic is $V_0 = n^3$.

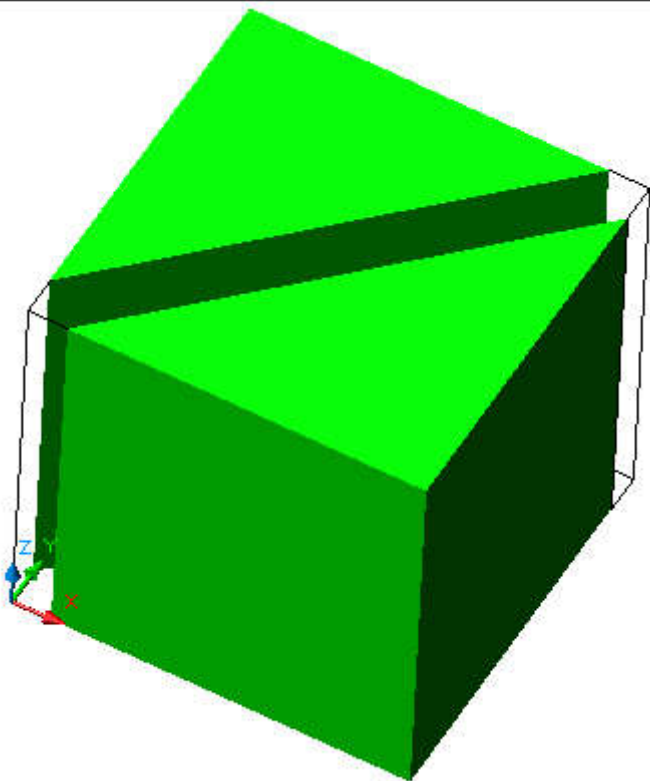


As we want to find the probability of the incident $A = \{|x - y| \geq 1, |y - z| \geq 1, |z - x| \geq 1\}$, we should find the volume of the region of points such that $|x - y| \geq 1$, $|y - z| \geq 1$, $|z - x| \geq 1$ and $0 \leq x, y, z \leq n$.

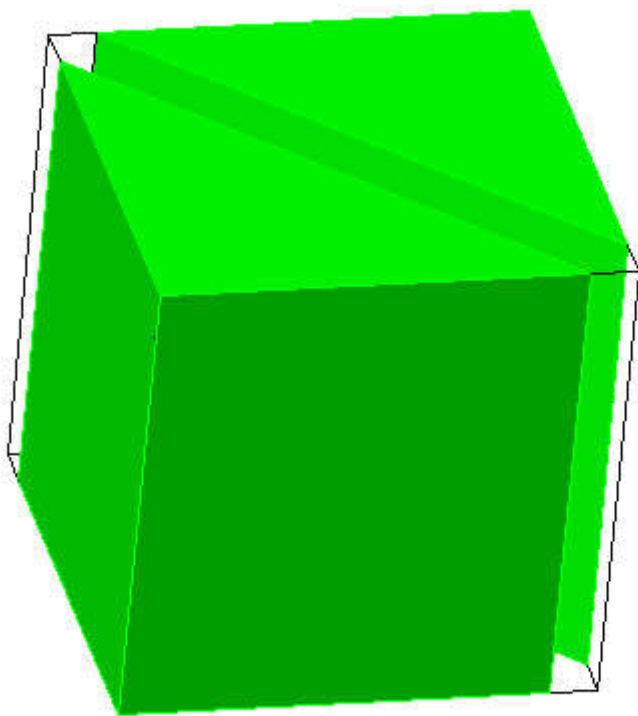
Now we will find the region $\{(x, y, z) \mid 0 \leq x, y, z \leq n, |x - y| \geq 1, |y - z| \geq 1, |z - x| \geq 1\}$.

The region can be generated by cutting off 3 slices corresponding to $|x - y| < 1$, $|y - z| < 1$, and $|z - x| < 1$, respectively, from the cubic.

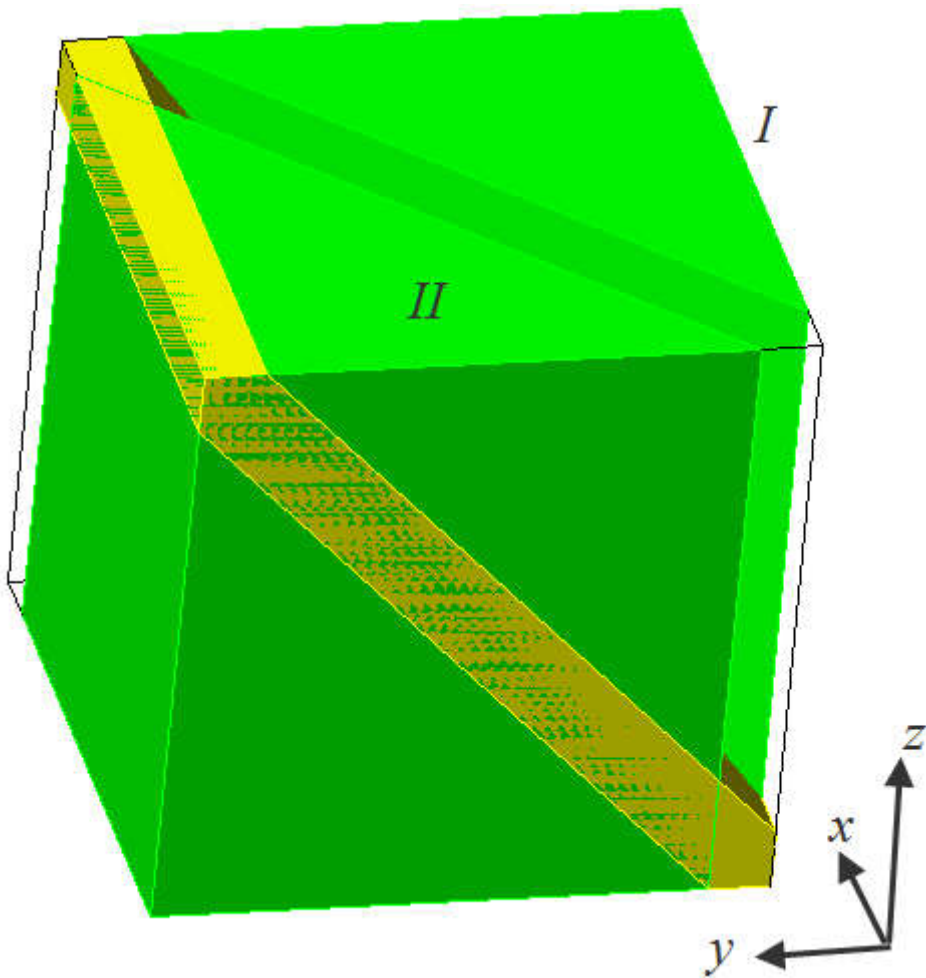
After cutting off a slice corresponding to $|x - y| < 1$, we get two triangular prisms, as shown in the figure.



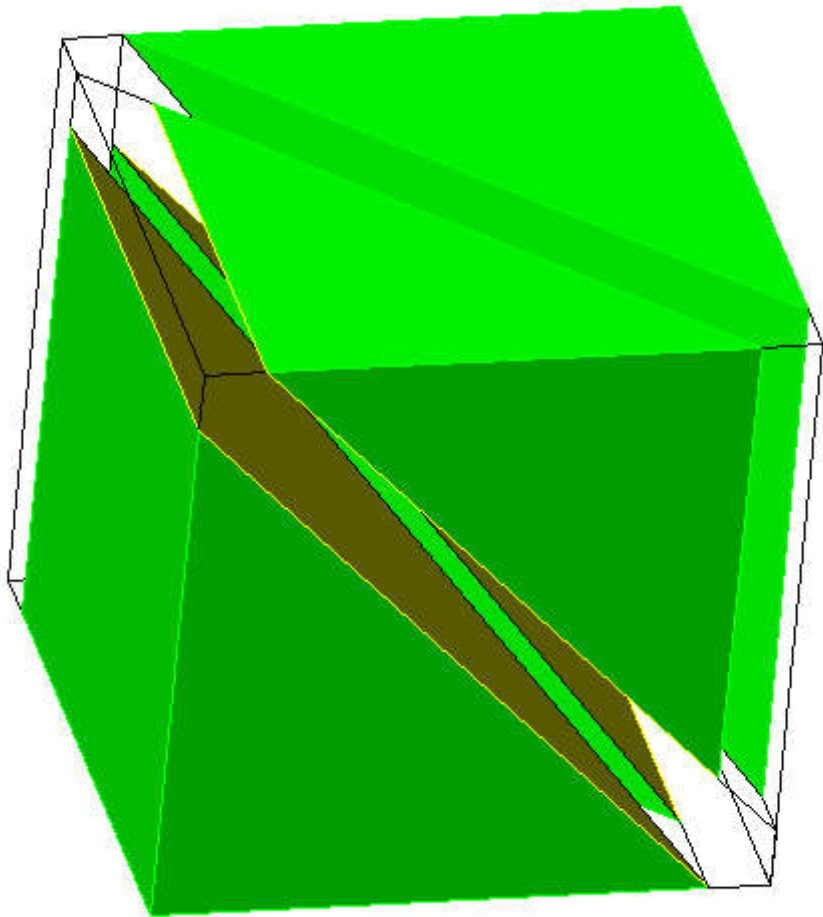
In order to observe the object clearly, we rotate the object by the z axis, as shown.



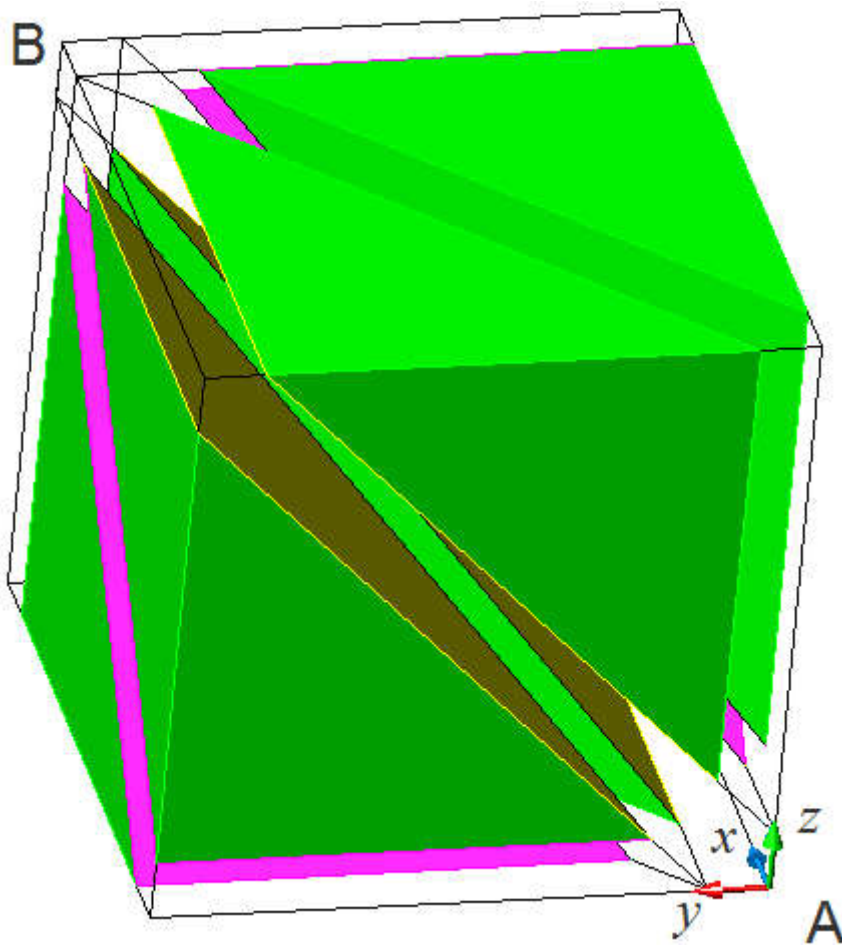
We can draw the slice corresponding to $|y - z| < 1$ on the object.



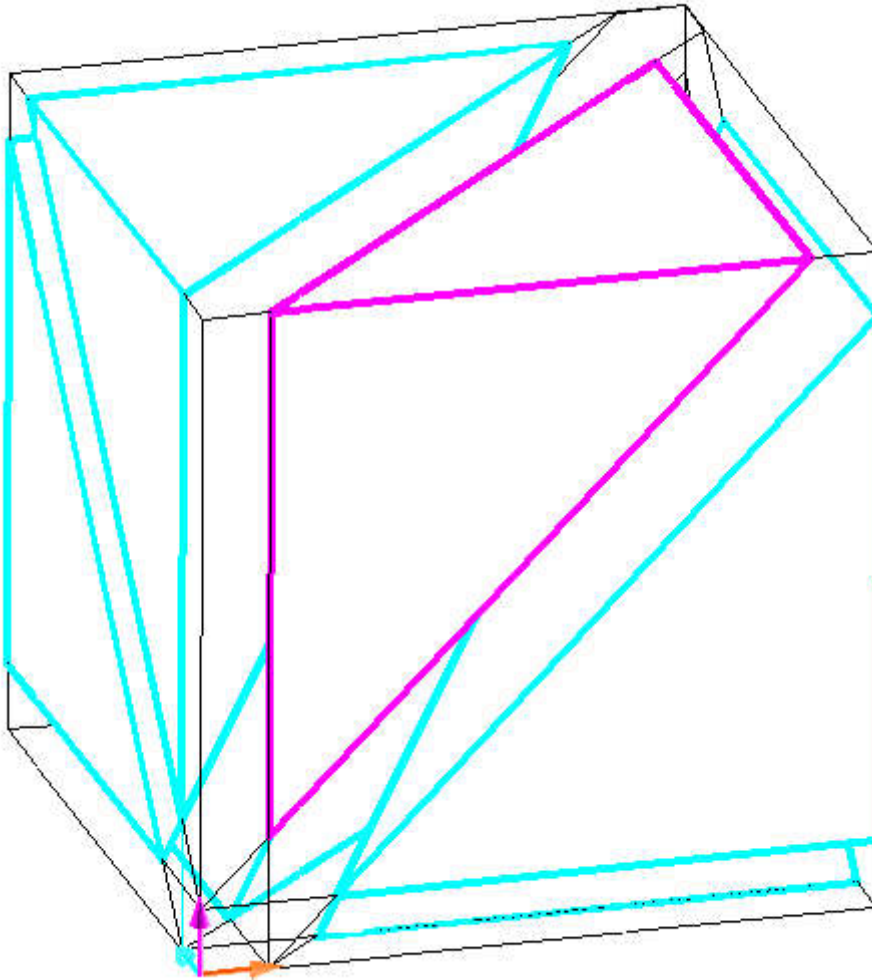
After cutting off the slice corresponding to $|y - z| < 1$, we have 4 pieces left.



After cutting off the slice corresponding to $|z - x| < 1$, we have 6 congruent triangular prisms.



Here we draw all the pictures in colors in order to explain the solution clearly. That does not mean that the students should do it in the examination. They can draw a figure with lines only, as shown below.



Every triangular pyramid has an altitude $n - 2$ and a base of isosceles right triangle with leg length $n - 2$, so the volume is $(n - 2)^3/6$. Then the volume of the region $\{(x, y, z) \mid 0 \leq x, y, z \leq n, |x - y| \geq 1, |y - z| \geq 1, |z - x| \geq 1\}$ is $V_A = 6 \times (n - 2)^3/6 = (n - 2)^3$.

So the probability of the incident A is $P(A) = \frac{V_A}{V_0} = \frac{(n - 2)^3}{n^3}$.

Then we can get the answer the same way as Solution I.

The answer is **(D) 10**.

If there is no choice for selection, we can also find the minimum value of the integer n if we do not substitute n by the possible values one by one.

Let $P(A) > 1/2$, i.e., $\frac{(n - 2)^3}{n^3} > \frac{1}{2}$, so $\frac{n - 2}{n} > \frac{1}{\sqrt[3]{2}}$, or $1 - \frac{2}{n} > \frac{1}{\sqrt[3]{2}}$, hence $n > \frac{2\sqrt[3]{2}}{\sqrt[3]{2} - 1}$.

Now we will estimate the value of $\frac{2\sqrt[3]{2}}{\sqrt[3]{2} - 1}$ without a calculator.

$$\text{Since } a^3 - 1 = (a - 1)(a^2 + a + 1), \text{ so } \frac{2\sqrt[3]{2}}{\sqrt[3]{2} - 1} = \frac{2\sqrt[3]{2} \times (\sqrt[3]{2}^2 + \sqrt[3]{2} + 1)}{(\sqrt[3]{2} - 1)(\sqrt[3]{2}^2 + \sqrt[3]{2} + 1)} =$$

$$\frac{2 \times (2 + \sqrt[3]{2}^2 + \sqrt[3]{2})}{\sqrt[3]{2}^3 - 1} = 2 \times (2 + \sqrt[3]{4} + \sqrt[3]{2}).$$

Now we would get the approximation of $\sqrt[3]{4}$ and $\sqrt[3]{2}$.

In order to avoid complicated computation, we get the approximation with one decimal digit only.

Estimation of $\sqrt[3]{2}$.

Since $1.5^3 = 2.25 \times 1.5 > 2$, so $1 < \sqrt[3]{2} < 1.5$.

The mean of 1 and 1.5 with one decimal digit is about 1.3.

As $1.3^3 = 1.69 \times 1.3 = 2.197 > 2$, so $1 < \sqrt[3]{2} < 1.3$.

The mean of 1 and 1.3 with one decimal digit is about 1.2.

As $1.2^3 = 1.44 \times 1.2 = 1.728 < 2$, so $1.2 < \sqrt[3]{2} < 1.3$.

Estimation of $\sqrt[3]{4}$.

As $\sqrt[3]{4} = \sqrt[3]{2}^2$, so $1.2^2 < \sqrt[3]{4} < 1.3^2$, then $1.24 < \sqrt[3]{4} < 1.69$.

As $1.5^3 = 2.25 \times 1.5 = 3.375 < 4$, so $1.5 < \sqrt[3]{4} < 1.69$.

The mean of 1.5 and 1.69 with one decimal digit is about 1.6.

As $1.6^3 = (16/10)^3 = (2^4/10)^3 = 2^{12}/10^3 = 4 \times 2^{10}/10^3 = 4 \times 1.024 > 4$, so $1.5 < \sqrt[3]{4} < 1.6$.

Then $2 \times (2 + 1.5 + 1.2) < 2 \times (2 + \sqrt[3]{4} + \sqrt[3]{2}) < 2 \times (2 + 1.6 + 1.3)$, i.e.,
 $9.4 < 2 \times (2 + \sqrt[3]{4} + \sqrt[3]{2}) < 9.8$,

As $n > 2 \times (2 + \sqrt[3]{4} + \sqrt[3]{2})$. So the minimal value of integer n is 10.

Appendix

This solution is motivated by the suggestive formula $\frac{(n-2)^3}{n^3}$.

The problem generalizes easily to k -dimensional real space $\mathbb{R}^k$. In the general k -dimensional case, we are asked to find the probability that a randomly chosen k -tuple $(x_1, \dots, x_k) \in [0, n]^k$ satisfies $|x_i - x_j| > 1$ for all $i \neq j$. To avoid repetition, let us say that $(x_1, \dots, x_k)$ is spaced-out if $|x_i - x_j| > 1$ for all $i \neq j$.

Let C be the k -dimensional hyper-cube of side length n :

$$C = [0, n]^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : 0 \leq x_i \leq n \text{ for all } i\}.$$

Then C has volume n^k . Let S be the set of spaced-out k -tuples $(x_1, \dots, x_k)$. The desired probability is $\text{Vol}(S)/n^k$.

The set of k -tuples $(x_1, \dots, x_k)$ such that there exist distinct indices i, j such that $x_i = x_j$ has volume 0, so we may restrict our attention to k -tuples such that $x_i \neq x_j$ for all $i \neq j$.

Further, the condition that $(x_1, \dots, x_k)$ is spaced-out is "invariant upon permuting the indices"; in other words, if σ is a permutation of the set of indices $\{1, \dots, k\}$, then $(x_1, \dots, x_k)$ is spaced-out if and only if $(x_{\sigma(1)}, \dots, x_{\sigma(k)})$ is spaced-out. Therefore, we may consider the set of spaced-out k -tuples $(x_1, \dots, x_k)$ which additionally satisfy $x_1 < \dots < x_k$. Let us denote this set by T . This condition is equivalent to

$$0 \leq x_1 < x_2 - 1 < \dots < x_i - (i-1) < \dots < x_k - (k-1) \leq n - (k-1).$$

Let us choose new variables $y_i = x_i - (i-1)$ for $i = 1, \dots, k$. This change of variables is just a translation of each $(x_1, \dots, x_k)$ by the vector $(0, 1, \dots, k-1)$; in the above solutions, it corresponds to taking the 6 tetrahedrons and gluing them together to form a cube.

We now compute the volume of the set of $(y_1, \dots, y_k) \in [0, n - (k-1)]^k$ which satisfy $y_1 < \dots < y_k$. As above, we can disregard any $(y_1, \dots, y_k)$ such that $y_i = y_j$ for some $i \neq j$. Given any $(y_1, \dots, y_k)$ such that $y_i \neq y_j$ for all $i \neq j$, there exists exactly one permutation σ of the indices $\{1, \dots, k\}$ such that

$y_{\sigma(1)} < \dots < y_{\sigma(k)}$. Since there are $k!$ permutations of $\{1, \dots, k\}$, the desired volume is equal to $\frac{1}{k!}$ times the volume of the k -dimensional hyper-cube of side length $n - (k-1)$, which is $\frac{1}{k!}(n - (k-1))^k$. Hence T has volume $\frac{1}{k!}(n - (k-1))^k$ as well and S has volume $(n - (k-1))^k$. Hence the desired probability is $\frac{(n - (k-1))^k}{n^k}$.

See Also

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