2013 AMC 10B Problems/Problem 1

Problem

What is $\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6}$?

$\textbf{(A)}\ -1 \qquad \textbf{(B)}\ \frac{5}{36} \qquad \textbf{(C)}\ \frac{7}{12} \qquad \textbf{(D)}\ \frac{49}{20} \qquad \textbf{(E)}\ \frac{43}{3}$

Solution

This expression is equivalent to $\frac{12}{9} - \frac{9}{12} = \frac{16}{12} - \frac{9}{12} = \boxed{\textbf{(C)}\ \frac{7}{12}}$

See Also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by First Problem	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B Problems/Problem 2

The following problem is from both the 2013 AMC 12B #2 and 2013 AMC 10B #2, so both problems redirect to this page.

Problem

Mr. Green measures his rectangular garden by walking two of the sides and finds that it is 15 steps by 20 steps. Each of Mr. Green's steps is 2 feet long. Mr. Green expects a half a pound of potatoes per square foot from his garden. How many pounds of potatoes does Mr. Green expect from his garden?

$\textbf{(A)}\ 600 \qquad \textbf{(B)}\ 800 \qquad \textbf{(C)}\ 1000 \qquad \textbf{(D)}\ 1200 \qquad \textbf{(E)}\ 1400$

Solution

Since each step is 2 feet, his garden is 30 by 40 feet. Thus, the area of $30(40) = 1200$ square feet. Since he is expecting $\frac{1}{2}$ of a pound per square foot, the total amount of potatoes expected is

$1200 \times \frac{1}{2} = \boxed{\textbf{(A)}\ 600}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2013 AMC 12B Problems/Problem 1

The following problem is from both the 2013 AMC 12B #1 and 2013 AMC 10B #3, so both problems redirect to this page.

Problem

On a particular January day, the high temperature in Lincoln, Nebraska, was 16 degrees higher than the low temperature, and the average of the high and low temperatures was 3°. In degrees, what was the low temperature in Lincoln that day?

$\textbf{(A)}\ -13 \qquad \textbf{(B)}\ -8 \qquad \textbf{(C)}\ -5 \qquad \textbf{(D)}\ -3 \qquad \textbf{(E)}\ 11$

Solution

Let L be the low temperature. The high temperature is $L+16$. The average is $\frac{L+(L+16)}{2}=3$. Solving for L, we get $L=\boxed{\textbf{(C)}\ -5}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by First Question	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2013 AMC 12B Problems/Problem 3

The following problem is from both the 2013 AMC 12B #3 and 2013 AMC 10B #4, so both problems redirect to this page.

Problem

When counting from 3 to 201, 53 is the 51^{st} number counted. When counting backwards from 201 to 3, 53 is the n^{th} number counted. What is n?

$\textbf{(A)}\ 146 \qquad \textbf{(B)}\ 147 \qquad \textbf{(C)}\ 148 \qquad \textbf{(D)}\ 149 \qquad \textbf{(E)}\ 150$

Solution

Note that n is equal to the number of integers between 53 and 201, inclusive. Thus,
$n = 201 - 53 + 1 = \boxed{\textbf{(D)}\ 149}$

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 10B Problems/Problem 5

Problem

Positive integers a and b are each less than 6. What is the smallest possible value for $2 \cdot a - a \cdot b$?

$\textbf{(A)}\ -20\qquad\textbf{(B)}\ -15\qquad\textbf{(C)}\ -10\qquad\textbf{(D)}\ 0\qquad\textbf{(E)}\ 2$

Solution

Factoring the equation gives $a(2-b)$. From this we can see that to obtain the least possible value, $2-b$ should be negative, and should be as small as possible. To do so, b should be maximized. Because $2-b$ is negative, we should maximize the positive value of a as well. The maximum values of both a and b are 5, so the answer is $5(2-5) = \boxed{\textbf{(B)}\ -15}$.

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B Problems/Problem 5

The following problem is from both the 2013 AMC 12B #5 and 2013 AMC 10B #6, so both problems redirect to this page.

Problem

The average age of 33 fifth-graders is 11. The average age of 55 of their parents is 33. What is the average age of all of these parents and fifth-graders?

$\textbf{(A)}\ 22 \qquad \textbf{(B)}\ 23.25 \qquad \textbf{(C)}\ 24.75 \qquad \textbf{(D)}\ 26.25 \qquad \textbf{(E)}\ 28$

Solution

The sum of the ages of the fifth graders is $33*11$, while the sum of the ages of the parents is $55*33$. Therefore, the total sum of all their ages must be 2178, and given $33+55=88$ people in total, their average age is $\frac{2178}{88}=\frac{99}{4}=\boxed{\textbf{(C)}\ 24.75}$.

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 10B Problems/Problem 7

Problem

Six points are equally spaced around a circle of radius 1. Three of these points are the vertices of a triangle that is neither equilateral nor isosceles. What is the area of this triangle?

$\textbf{(A)}\ \frac{\sqrt{3}}{3}\qquad\textbf{(B)}\ \frac{\sqrt{3}}{2}\qquad\textbf{(C)}\ 1\qquad\textbf{(D)}\ \sqrt{2}\qquad\textbf{(E)}\ 2$

Solution

If there are no two points on the circle that are adjacent, then the triangle would be equilateral. If the three points are all adjacent, it would be isosceles. Thus, the only possibility is two adjacent points and one point two away. Because one of the sides of this triangle is the diameter, the opposite angle is a right angle. Also, because the two adjacent angles are one sixth of the circle apart, the angle opposite them is thirty degrees. This is a $30-60-90$ triangle. If the original six points are connected, a regular hexagon is created. This hexagon consists of six equilateral triangles, so the radius is equal to one of its side lengths. The radius is 1, so the side opposite the thirty degree angle in the triangle is also 1. From rules with $30-60-90$ triangles, the area is $1\cdot\sqrt{3}/2=\boxed{\textbf{(B)}\ \frac{\sqrt{3}}{2}}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Geometry Problems

2013 AMC 12B Problems/Problem 4

The following problem is from both the 2013 AMC 12B #4 and 2013 AMC 10B #8, so both problems redirect to this page.

Problem

Ray's car averages 40 miles per gallon of gasoline, and Tom's car averages 10 miles per gallon of gasoline. Ray and Tom each drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline?
$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 16 \qquad \textbf{(C)}\ 25 \qquad \textbf{(D)}\ 30 \qquad \textbf{(E)}\ 40$

Solution

Let both Ray and Tom drive 40 miles. Ray's car would require $\frac{40}{40}=1$ gallon of gas and Tom's car would require $\frac{40}{10}=4$ gallons of gas. They would have driven a total of $40+40=80$ miles, on $1+4=5$ gallons of gas, for a combined rate of $\frac{80}{5}=\boxed{\textbf{(B)}\ 16}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2013_AMC_12B_Problems/Problem_4&oldid=67238"

2013 AMC 10B Problems/Problem 9

Problem

Three positive integers are each greater than 1, have a product of 27000, and are pairwise relatively prime. What is their sum?

$\textbf{(A)}\ 100\qquad\textbf{(B)}\ 137\qquad\textbf{(C)}\ 156\qquad\textbf{(D)}\ 160\qquad\textbf{(E)}\ 165$

Solution

The prime factorization of 27000 is $2^3*3^3*5^3$. These three factors are pairwise relatively prime, so the sum is $2^3+3^3+5^3=8+27+125=\boxed{\textbf{(D)}\ 160}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 10B Problems/Problem 10

Problem

A basketball team's players were successful on 50% of their two-point shots and 40% of their three-point shots, which resulted in 54 points. They attempted 50% more two-point shots than three-point shots. How many three-point shots did they attempt?

$\textbf{(A)}\ 10\qquad\textbf{(B)}\ 15\qquad\textbf{(C)}\ 20\qquad\textbf{(D)}\ 25\qquad\textbf{(E)}\ 30$

Solution

Call x the number of two point shots attempted and y the number of three point shots attempted. Because each two point shot is worth two points and the team made 50% and each three point shot is worth 3 points and the team made 40%, $0.5(2x)+0.4(3y)=54$ or $x+1.2y=54$. Because the team attempted 50% more two point shots then threes, $x=1.5y$. Substituting $1.5y$ for x in the first equation gives $1.5y+1.2y=54$, which equals $2.7y=54$ so $y=\boxed{\textbf{(C)}\ 20}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B Problems/Problem 6

The following problem is from both the 2013 AMC 12B #6 and 2013 AMC 10B #11, so both problems redirect to this page.

Problem

Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 8$

Solution

If we complete the square after bringing the x and y terms to the other side, we get $(x-5)^2 + (y+3)^2 = 0$. Squares of real numbers are nonnegative, so we need both $(x-5)^2$ and $(y+3)^2$ to be 0, which only happens when $x = 5$ and $y = -3$. Therefore, $x + y = 5 + (-3) = \boxed{\textbf{(B)}\ 2}$.

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2013_AMC_12B_Problems/Problem_6&oldid=67239"

Categories: Introductory Algebra Problems | Algebraic Manipulations Problems

2013 AMC 10B Problems/Problem 12

Problem

Let S be the set of sides and diagonals of a regular pentagon. A pair of elements of S are selected at random without replacement. What is the probability that the two chosen segments have the same length?

$\textbf{(A)}\ \frac{2}{5} \qquad\textbf{(B)}\ \frac{4}{9} \qquad\textbf{(C)}\ \frac{1}{2} \qquad\textbf{(D)}\ \frac{5}{9} \qquad\textbf{(E)}\ \frac{4}{5}$

Solution

In a regular pentagon, there are 5 sides with the same length, and 5 diagonals with the same length. Picking an element at random will leave 4 elements with the same length as the element picked, with 9 total elements remaining. Therefore, the probability is $\boxed{\textbf{(B)}\ \frac{4}{9}}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B Problems/Problem 7

The following problem is from both the 2013 AMC 12B #7 and 2013 AMC 10B #13, so both problems redirect to this page.

Problem

Jo and Blair take turns counting from 1 to one more than the last number said by the other person. Jo starts by saying "1", so Blair follows by saying "$1, 2$". Jo then says "$1, 2, 3$", and so on. What is the 53^{rd} number said?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 8$

Solution

We notice that the number of numbers said is incremented by one each time; that is, Jo says one number, then Blair says two numbers, then Jo says three numbers, etc. Thus, after nine "turns," $1+2+3+4+5+6+7+8+9=45$ numbers have been said. In the tenth turn, the eighth number will be the 53rd number said, as $53-45=8$. Since we're starting from 1 each time, the 53rd number said will be $\boxed{\textbf{(E)}\ 8}$.

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 12	Followed by Problem 14
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2013 AMC 10B Problems/Problem 14

Contents

- 1 Problem
- 2 Solution
- 3 Alternate Solution
- 4 See also

Problem

Define $a\clubsuit b = a^2b - ab^2$. Which of the following describes the set of points (x, y) for which $x\clubsuit y = y\clubsuit x$?

$\textbf{(A)}$ a finite set of points
$\textbf{(B)}$ one line
$\textbf{(C)}$ two parallel lines
$\textbf{(D)}$ two intersecting lines
$\textbf{(E)}$ three lines

Solution

$x\clubsuit y = x^2y - xy^2$ and $y\clubsuit x = y^2x - yx^2$. Therefore, we have the equation $x^2y - xy^2 = y^2x - yx^2$ Factoring out a -1 gives $x^2y - xy^2 = -(x^2y - xy^2)$ Factoring both sides further, $xy(x - y) = -xy(x - y)$. It follows that if $x = 0$, $y = 0$, or $(x - y) = 0$, both sides of the equation equal 0. By this, there are 3 lines ($x = 0$, $y = 0$, or $x = y$) so the answer is $\boxed{\textbf{(E)}\text{ three lines}}$.

Alternate Solution

Following from the previous solution, $x^2y - xy^2 = y^2x - yx^2$. Then, $2x^2y - 2xy^2 = 0$. Factoring, $2xy(x - y) = 0$. Now, the solutions are obviously $x = 0$, $y = 0$, or $x = y$, which each correspond to a line. Thus, the answer is $\boxed{\textbf{(E)}\text{ three lines}}$.

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 10B Problems/Problem 15

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See also

Problem

A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ \frac{\sqrt{6}}{2} \qquad \textbf{(C)}\ \sqrt{3} \qquad \textbf{(D)}\ 2 \qquad \textbf{(E)}\ \frac{3\sqrt{2}}{2}$

Solution 1

Using the formulas for area of a regular triangle $(\frac{s^2\sqrt{3}}{4})$ and regular hexagon $(\frac{3s^2\sqrt{3}}{2})$ and plugging $\frac{a}{3}$ and $\frac{b}{6}$ into each equation, you find that $\frac{a^2\sqrt{3}}{36} = \frac{b^2\sqrt{3}}{24}$. Simplifying this, you get $\frac{a}{b} = \boxed{\textbf{(B)} \frac{\sqrt{6}}{2}}$

Solution 2

The regular hexagon can be broken into 6 small equilateral triangles, each of which is similar to the big equilateral triangle. The big triangle's area is 6 times the area of one of the little triangles. Therefore each side of the big triangle is $\sqrt{6}$ times the side of the small triangle. The desired ratio is $\frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2} \Rightarrow (B)$.

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Categories: Introductory Geometry Problems | Area Problems

2013 AMC 10B Problems/Problem 16

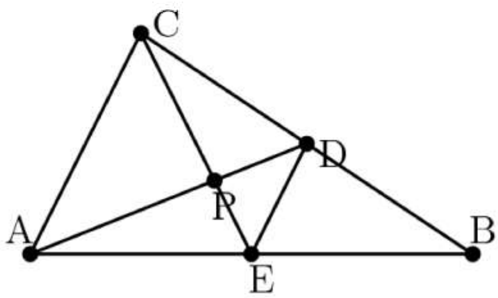
Contents

- 1 Problem
- 2 Solution
  - 2.1 Solution 1
  - 2.2 Solution 2
  - 2.3 Solution 3
- 3 See also

Problem

In triangle ABC, medians AD and CE intersect at P, $PE=1.5$, $PD=2$, and $DE=2.5$. What is the area of $AEDC$?

$\textbf{(A)}13 \qquad \textbf{(B)}13.5 \qquad \textbf{(C)}14 \qquad \textbf{(D)}14.5 \qquad \textbf{(E)}15$



Solution

Solution 1

Let us use mass points: Assign B mass 1. Thus, because E is the midpoint of AB, A also has a mass of 1. Similarly, C has a mass of 1. D and E each have a mass of 2 because they are between B and C and A and B respectively. Note that the mass of D is twice the mass of A, so AP must be twice as long as PD. PD has length 2, so AP has length 4 and AD has length 6. Similarly, CP is twice PE and $PE=1.5$, so $CP=3$ and $CE=4.5$. Now note that triangle PED is a $3-4-5$ right triangle with the right angle DPE. Since the diagonals of quadrilaterals $AEDC$, AD and CE, are perpendicular, the area of $AEDC$ is $\frac{6\times 4.5}{2} = \boxed{\textbf{(B)}13.5}$

Solution 2

Note that triangle DPE is a right triangle, and that the four angles that have point P are all right angles. Using the fact that the centroid (P) divides each median in a $2:1$ ratio, $AP=4$ and $CP=3$. Quadrilateral $AEDC$ is now just four right triangles. The area is $\frac{4\cdot 1.5+4\cdot 3+3\cdot 2+2\cdot 1.5}{2}=\boxed{\textbf{(B)}13.5}$

Solution 3

From the solution above, we can find that the lengths of the diagonals are 6 and 4.5. Now, since the diagonals of AEDC are perpendicular, we use the area formula to find that the total area is $\frac{6\times 4.5}{2} = \frac{27}{2} = 13.5 = \boxed{B}$

See also

2013 AMC 12B Problems/Problem 10

The following problem is from both the 2013 AMC 12B #10 and 2013 AMC 10B #17, so both problems redirect to this page.

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See also

Problem

Alex has 75 red tokens and 75 blue tokens. There is a booth where Alex can give two red tokens and receive in return a silver token and a blue token, and another booth where Alex can give three blue tokens and receive in return a silver token and a red token. Alex continues to exchange tokens until no more exchanges are possible. How many silver tokens will Alex have at the end?

$\textbf{(A)}\ 62 \qquad \textbf{(B)}\ 82 \qquad \textbf{(C)}\ 83 \qquad \textbf{(D)}\ 102 \qquad \textbf{(E)}\ 103$

Solution 1

We can approach this problem by assuming he goes to the red booth first. You start with 75R and 75B and at the end of the first booth, you will have 1R and 112B and 37S. We now move to the blue booth, and working through each booth until we have none left, we will end up with:1R, 2B and 103S. So, the answer is $\boxed{\textbf{(E)}103}$

Solution 2

Let x denote the number of visits to the first booth and y denote the number of visits to the second booth. Then we can describe the quantities of his red and blue coins as follows:

$$R(x,y) = -2x + y + 75$$

$$B(x,y) = x - 3y + 75$$

There are no legal exchanges when he has fewer than 2 red coins and fewer than 3 blue coins, namely when he has 1 red coin and 2 blue coins. We can then create a system of equations:

$$1 = -2x + y + 75$$

$$2 = x - 3y + 75$$

Solving yields $x = 59$ and $y = 44$. Since he gains one silver coin per visit to each booth, he has $x + y = 44 + 59 = \boxed{\textbf{(E)}103}$ silver coins in total.

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 10B Problems/Problem 18

Problem

The number 2013 has the property that its units digit is the sum of its other digits, that is $2+0+1=3$. How many integers less than 2013 but greater than 1000 share this property?

$\textbf{(A)}\ 33\qquad\textbf{(B)}\ 34\qquad\textbf{(C)}\ 45\qquad\textbf{(D)}\ 46\qquad\textbf{(E)}\ 58$

Solution

First, note that the only integer $2000 \le x < 2013$ is 2002. Now let's look at all numbers x where $1000 < x < 2000$ Let the hundreds digit be 0. Then, the tens and units digit can be $01, 12, 23, \ldots, 89$, which is 9 possibilities. We notice as the hundreds digit goes up by one, the number of possibilities goes down by one. Thus, the number of integers is $1+2+3+4+5+6+7+8+9+1 = \boxed{\textbf{(D)} 46}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 10B Problems/Problem 19

Contents

- 1 Problem
- 2 Solutions
  - 2.1 Solution 1
  - 2.2 Solution 2
  - 2.3 Solution 3
  - 2.4 Solution 4
- 3 See also

Problem

The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

$\textbf{(A)}\ -7-4\sqrt{3} \qquad \textbf{(B)}\ -2-\sqrt{3} \qquad \textbf{(C)}\ -1 \qquad \textbf{(D)}\ -2+\sqrt{3} \qquad \textbf{(E)}\ -7+4\sqrt{3}$

Solutions

Solution 1

It is given that $ax^2+bx+c=0$ has 1 real root, so the discriminant is zero, or $b^2=4ac$. Because a, b, c are in arithmetic progression, $b-a=c-b$, or $b=\frac{a+c}{2}$. We need to find the unique root, or $-\frac{b}{2a}$ (discriminant is 0). From $b^2=4ac$, we can get $-\frac{b}{2a}=-\frac{2c}{b}$. Ignoring the negatives, we have $\frac{2c}{b}=\frac{2c}{\frac{a+c}{2}}=\frac{4c}{a+c}=\frac{1}{\frac{1}{\frac{4c}{a+c}}}=\frac{1}{\frac{a+c}{4c}}=\frac{1}{\frac{a}{4c}+\frac{1}{4}}$. Fortunately, finding $\frac{a}{c}$ is not very hard. Plug in $b=\frac{a+c}{2}$ to $b^2=4ac$, we have $a^2+2ac+c^2=16ac$, or $a^2-14ac+c^2=0$, and dividing by c^2 gives $(\frac{a}{c})^2-14(\frac{a}{c})+1=0$, so $\frac{a}{c}=\frac{14\pm\sqrt{192}}{2}=7\pm4\sqrt{3}$. But $7-4\sqrt{3}<1$, violating the assumption that $a\ge c$. Therefore, $\frac{a}{c}=7+4\sqrt{3}$. Plugging this in, we have $\frac{1}{\frac{a}{4c}+\frac{1}{4}}=\frac{1}{2+\sqrt{3}}=2-\sqrt{3}$. But we need the negative of this, so the answer is $\boxed{\textbf{(D)}}$.

Solution 2

Note that we can divide the polynomial by a to make the leading coefficient 1 since dividing does not change the roots or the fact that the coefficients are in an arithmetic sequence. Also, we know that there is exactly one root so this equation must be of the form $(x-r)^2 = x^2 - 2rx + r^2$ where $1 \geq -2r \geq r^2 \geq 0$. We now use the fact that the coefficients are in an arithmetic sequence. Note that in any arithmetic sequence, the average is equal to the median. Thus, $r^2 + 1 = -4r$ and $r = -2 \pm \sqrt{3}$. Since $1 > r^2$, we easily see that $|r|$ has to be between 1 and 0. Thus, we can eliminate $-2-\sqrt{3}$ and are left with $\boxed{\textbf{(D)}-2+\sqrt{3}}$ as the answer.

Solution 3

Given that $ax^2+bx+c=0$ has only 1 real root, we know that the discriminant must equal 0, or that $b^2=4ac$. Because the discriminant equals 0, we have that the root of the quadratic is $r=\frac{-b}{2a}$. We are also given that the coefficients of the quadratic are in arithmetic progression, where $a\geq b\geq c\geq 0$. Letting the arbitrary difference equal variable d, we have that $a=b+d$ and that $c=b-d$. Plugging those two equations into $b^2=4ac$, we have $b^2=4(b^2-d^2)=4b^2-4d^2$ which yields $3b^2=4d^2$. Isolating d, we have $d = \frac{b\sqrt{3}}{2}$. Substituting that in for d in $a = b + d$, we get $a = b + \frac{b\sqrt{3}}{2} = b(1 + \frac{\sqrt{3}}{2})$. Once again, substituting that in for a in $r = \frac{-b}{2a}$, we have $r = \frac{-b}{2b(1+\frac{\sqrt{3}}{2})} = \frac{-1}{2+\sqrt{3}} = -2 + \sqrt{3}$. The answer is:

$$\boxed{\textbf{(D)}}.$$

Solution 4

Let the double root be r. Then by the arithmetic progression and Vieta's,

$$\begin{aligned} a - b &= b - c \\ 1 - \frac{b}{a} &= \frac{b}{a} - \frac{c}{a} \\ 1 + 2r &= -2r - r^2 \\ r^2 + 4r + 1 &= 0 \\ r &= -2 \pm \sqrt{3} \end{aligned}$$

We see $0 \leq b \leq a \Rightarrow 0 \leq \frac{b}{a} \leq 1$, and so we want $0 \leq -2r \leq 1$. Note that since $0 \leq -2(-2-\sqrt{3}) = 4 + 2\sqrt{3} \geq 1$ and $0 \leq -2(-2+\sqrt{3}) = 4 - 2\sqrt{3} \leq 1$, we can conclude that $r = -2 + \sqrt{3}$, so the answer is:

$$\boxed{\textbf{(D)}}.$$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 12B Problems/Problem 15

The following problem is from both the 2013 AMC 12B #15 and 2013 AMC 10B #20, so both problems redirect to this page.

Problem

The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!...a_m!}{b_1!b_2!...b_n!},$$

where $a_1 \ge a_2 \ge \cdots \ge a_m$ and $b_1 \ge b_2 \ge \cdots \ge b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

The prime factorization of 2013 is $61\cdot11\cdot3$. To have a factor of 61 in the numerator and to minimize a_1, a_1 must equal 61. Now we notice that there can be no prime p which is not a factor of 2013 such that $b_1<p<61$, because this prime will not be represented in the denominator, but will be represented in the numerator. The highest p less than 61 is 59, so there must be a factor of 59 in the denominator. It follows that $b_1=59$ (to minimize b_1 as well), so the answer is $|61-59| = \boxed{\textbf{(B) }2}$. One possible way to express 2013 with $(a_1,b_1) = (61,59)$ is

$$2013 = \frac{61!\cdot19!\cdot11!}{59!\cdot20!\cdot10!}.$$

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2013_AMC_12B_Problems/Problem_15&oldid=73942"

Category: Introductory Number Theory Problems

2013 AMC 12B Problems/Problem 14

The following problem is from both the 2013 AMC 12B #14 and 2013 AMC 10B #21, so both problems redirect to this page.

Problem

Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N ?

$\textbf{(A)}\ 55 \qquad \textbf{(B)}\ 89 \qquad \textbf{(C)}\ 104 \qquad \textbf{(D)}\ 144 \qquad \textbf{(E)}\ 273$

Solution

Let the first two terms of the first sequence be x_1 and x_2 and the first two of the second sequence be y_1 and y_2. Computing the seventh term, we see that $5x_1 + 8x_2 = 5y_1 + 8y_2$. Note that this means that x_1 and y_1 must have the same value modulo 8. To minimize, let one of them be 0; WLOG assume that $x_1 = 0$. Thus, the smallest possible value of y_1 is 8; since the sequences are non-decreasing $y_2 \geq 8$. To minimize, let $y_2 = 8$. Thus, $5y_1 + 8y_2 = 40 + 64 = \boxed{\textbf{(C)}\ 104}$.

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 20	Followed by Problem 22
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 10B Problems/Problem 22

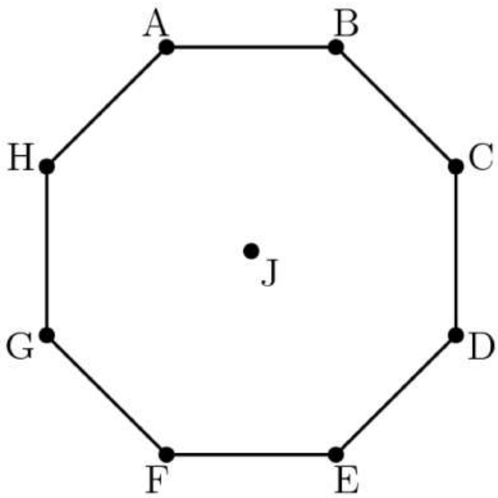
Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See also

Problem

The regular octagon $ABCDEFGH$ has its center at J. Each of the vertices and the center are to be associated with one of the digits 1 through 9, with each digit used once, in such a way that the sums of the numbers on the lines AJE, BJF, CJG, and DJH are all equal. In how many ways can this be done?

$\textbf{(A)}\ 384 \qquad \textbf{(B)}\ 576 \qquad \textbf{(C)}\ 1152 \qquad \textbf{(D)}\ 1680 \qquad \textbf{(E)}\ 3456$



Solution 1

First of all, note that J must be 1, 5, or 9 to preserve symmetry. We also notice that $A+E=B+F=C+G=D+H$.

Assume that $J=1$. Thus the pairs of vertices must be 9 and 2, 8 and 3, 7 and 4, and 6 and 5. There are $4!=24$ ways to assign these to the vertices. Furthermore, there are $2^4=16$ ways to switch them (i.e. do $2\ 9$ instead of $9\ 2$).

Thus, there are $16(24)=384$ ways for each possible J value. There are 3 possible J values that still preserve symmetry: $384(3)=\boxed{\textbf{(C)}\ 1152}$

Solution 2

As in solution 1, J must be 1, 5, or 9 giving us 3 choices. Additionally $A+E=B+F=C+G=D+H$. This means once we choose J there are 8 remaining choices. Going clockwise from A we count, 8 possibilities for A. Choosing A also determines E which leaves 6 choices for B, once B is chosen it also determines F leaving 4 choices for C. Once C is chosen it determines G leaving 2 choices for D. Choosing D determines H, exhausting the numbers. To get the answer we multiply $2*4*6*8*3=\boxed{\textbf{(C)}\ 1152}$.

See also

2013 AMC 12B Problems/Problem 19

The following problem is from both the 2013 AMC 12B #19 and 2013 AMC 10B #23, so both problems redirect to this page.

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See also

Problem

In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 21 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 27 \qquad \textbf{(E)}\ 30$

Solution 1

Since $\angle AFB = \angle ADB = 90^{\circ}$, quadrilateral $ABDF$ is cyclic. It follows that $\angle ADE = \angle ABF$. In addition, since $\angle AFB = \angle AED = 90$, triangles ABF and ADE are similar. It follows that $AF = (13)(\frac{4}{5}), BF = (13)(\frac{3}{5})$. By Ptolemy, we have $13DF + (5)(13)(\frac{4}{5}) = (12)(13)(\frac{3}{5})$. Cancelling 13, the rest is easy. We obtain $DF = \frac{16}{5} \implies 16 + 5 = 21 \implies \boxed{(B)}$

Solution 2

Using the similar triangles in triangle ADC gives $AE = \frac{48}{5}$ and $DE = \frac{36}{5}$. Quadrilateral $ABDF$ is cyclic, implying that $\angle B + \angle DFA$ = 180° . Therefore, $\angle B = \angle EFA$, and triangles AEF and ADB are similar. Solving the resulting proportion gives $EF = 4$. Therefore, $DF = ED - EF = \frac{16}{5}. \implies \boxed{(B)}$

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2013 AMC 10B Problems/Problem 24

Problem

A positive integer n is nice if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set $\{2010, 2011, 2012, \dots, 2019\}$ are nice?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

A positive integer with only four positive divisors has its prime factorization in the form of $a*b$, where a and b are both prime positive integers or c^3 where c is a prime. One can easily deduce that none of the numbers are even near a cube so that case is finished. We now look at the case of $a*b$. The four factors of this number would be 1, a, b, and ab. The sum of these would be $ab+a+b+1$, which can be factored into the form $(a+1)(b+1)$. Easily we can see that now we can take cases again.

Case 1: Either a or b is 2.

If this is true then we have to have that one of $(a+1)$ or $(b+1)$ is odd and that one is 3. The other is still even. So we have that in this case the only numbers that work are even multiples of 3 which are 2010 and 2016. So we just have to check if either $\frac{2016}{3} - 1$ or $\frac{2010}{3} - 1$ is a prime. We see that in this case none of them work.

Case 2: Both a and b are odd primes.

This implies that both $(a+1)$ and $(b+1)$ are even which implies that in this case the number must be divisible by 4. This leaves only 2012 and 2016. $2012 = 4*503$ so either $(a+1)$ or $(b+1)$ both has a factor of 2 or one has a factor of 4. If it was the first case, then a or b will equal 1. That means that either $(a+1)$ or $(b+1)$ has a factor of 4. That means that a or b is 502 which isn't a prime, so 2012 does not work. $2016 = 4*504$ so we have $(503+1)(3+1)$. 503 and 3 are both odd primes, so 2016 is a solution. Thus the answer is $\boxed{\textbf{(A)}\ 1}$.

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
Preceded by Problem 23	Followed by Problem 25
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2013_AMC_10B_Problems/Problem_24&oldid=75946"

Category: Introductory Number Theory Problems

2013 AMC 10B Problems/Problem 25

The following problem is from both the 2013 AMC 12B #23 and 2013 AMC 10B #25, so both problems redirect to this page.

Contents

- 1 Problem
- 2 Solution
- 3 Shortcut
- 4 See also

Problem

Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S. For example, if $N = 749$, Bernardo writes the numbers $10{,}444$ and $3{,}245$, and LeRoy obtains the sum $S = 13{,}689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 20 \qquad \textbf{(E)}\ 25$

Solution

First, we can examine the units digits of the number base 5 and base 6 and eliminate some possibilities.

Say that $N \equiv a \pmod{6}$

also that $N \equiv b \pmod{5}$

Substituting these equations into the question and setting the units digits of 2N and S equal to each other, it can be seen that $a = b$, and $b < 5$, (otherwise a and b always have different parities) so $N \equiv a \pmod{6}$, $N \equiv a \pmod{5}$, $\implies N = a \pmod{30}$, $0 \le a \le 4$

Therefore, N can be written as $30x + y$ and $2N$ can be written as $60x + 2y$

Keep in mind that y can be one of five choices: $0, 1, 2, 3,$ or 4, ; Also, we have already found which digits of y will add up into the units digits of $2N$.

Now, examine the tens digit, x by using $\mod 25$ and $\mod 36$ to find the tens digit (units digits can be disregarded because $y = 0, 1, 2, 3, 4$ will always work) Then we see that $N = 30x + y$ and take it $\mod 25$ and $\mod 36$ to find the last two digits in the base 5 and 6 representation.

$$N \equiv 30x \pmod{36}$$

$$N \equiv 30x \equiv 5x \pmod{25}$$

Both of those must add up to

$$2N \equiv 60x \pmod{100}$$

$(33 \ge x \ge 4)$

Now, since $y = 0, 1, 2, 3, 4$ will always work if x works, then we can treat x as a units digit instead of a tens digit in the respective bases and decrease the mods so that x is now the units digit.

$$N \equiv 6x \equiv x \pmod{5}$$

$$N \equiv 5x \pmod{6}$$

$$2N \equiv 6x \pmod{10}$$

Say that $x = 5m + n$ (m is between 0-6, n is 0-4 because of constraints on x) Then

$$N \equiv 5m + n \pmod 5$$

$$N \equiv 25m + 5n \pmod 6$$

$$2N \equiv 30m + 6n \pmod{10}$$

and this simplifies to

$$N \equiv n \pmod 5$$

$$N \equiv m + 5n \pmod 6$$

$$2N \equiv 6n \pmod{10}$$

From inspection, when

$n = 0, m = 6$

$n = 1, m = 6$

$n = 2, m = 2$

$n = 3, m = 2$

$n = 4, m = 4$

This gives you 5 choices for x, and 5 choices for y, so the answer is $5 * 5 = \boxed{\textbf{(E) } 25}$

Shortcut

Notice that there are exactly $1000 - 100 = 900 = 5^2 \cdot 6^2$ possible values of n. This means, from $100 \leq n \leq 999$, every possible combination of 2 digits will happen exactly once. We know that $n = 900, 901, 902, 903, 904$ works because $900 \equiv \ldots 00_5 \equiv \ldots 00_6$.

We know for sure that the units digit will add perfectly every 30 added or subtracted, because $\text{lcm } 5, 6 = 30$. So we only have to care about cases of n every 30 subtracted. In each case, $2n$ subtracts 6/adds 4, n_5 subtracts 1 and n_6 adds 1 for the 10's digit.

$$\begin{array}{l} \textbf{5} \quad 0\ 4\ 3\ 2\ 1\ 0\ 4\ 3\ 2\ 1\ 0\ 4\ 3\ 2\ 1\ 0\ 4\ 3\ 2\ 1\ 0\ 4\ 3\ 2\ 1\ 0\ 4\ 3\ 2\ 1 \\ \\ \textbf{6} \quad 0\ 1\ 2\ 3\ 4\ 5\ 0\ 1\ 2\ 3\ 4\ 5\ 0\ 1\ 2\ 3\ 4\ 5\ 0\ 1\ 2\ 3\ 4\ 5\ 0\ 1\ 2\ 3\ 4\ 5 \\ \\ \textbf{10}\ 0\ 4\ 8\ 2\ 6\ 0\ 4\ 8\ 2\ 6\ 0\ 4\ 8\ 2\ 6\ 0\ 4\ 8\ 2\ 6\ 0\ 4\ 8\ 2\ 6\ 0\ 4\ 8\ 2\ 6 \end{array}$$

As we can see, there are 5 cases, including the original, that work. These are highlighted in red. So, thus, there are 5 possibilities for each case, and $5 \cdot 5 = \boxed{\textbf{(E) } 25}$.

See also