

2017 AMC 10A Problems/Problem 1

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4
- 6 See Also

Problem

What is the value of $(2(2(2(2(2(2+1)+1)+1)+1)+1)+1)$?

(A) 70 (B) 97 (C) 127 (D) 159 (E) 729

Solution 1

Notice this is the term a_6 in a recursive sequence, defined recursively as $a_1 = 3, a_n = 2a_{n-1} + 1$. Thus:

$$a_2 = 3 * 2 + 1 = 7.$$

$$a_3 = 7 * 2 + 1 = 15.$$

$$a_4 = 15 * 2 + 1 = 31.$$

$$a_5 = 31 * 2 + 1 = 63.$$

$$a_6 = 63 * 2 + 1 = \boxed{\text{(C) } 127}$$

Solution 2

Starting to compute the inner expressions, we see the results are $1, 3, 7, 15, \dots$. This is always 1 less than a power of 2 . The only admissible answer choice by this rule is thus $\boxed{\text{(C) } 127}$.

Solution 3

Working our way from the innermost parenthesis outwards and directly computing, we have $\boxed{\text{(C) } 127}$.

Solution 4

If you distribute this you get a sum of the powers of 2 . The largest power of 2 in the series is 64 , so the sum is $\boxed{\text{(C) } 127}$.

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by First Problem	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 10A Problems/Problem 2

Problem

Pablo buys popsicles for his friends. The store sells single popsicles for **\$1** each, **3**-popsicle boxes for **\$2** each, and **5**-popsicle boxes for **\$3**. What is the greatest number of popsicles that Pablo can buy with **\$8**?

- (A) 8 (B) 11 (C) 12 (D) 13 (E) 15

Solution

\$3 boxes give us the most popsicles/dollar, so we want to buy as many of those as possible. After buying **2**, we have **\$2** left. We cannot buy a third **\$3** box, so we opt for the **\$2** box instead (since it has a higher popsicles/dollar ratio than the **\$1** pack). We're now out of money. We bought $5 + 5 + 3 = 13$ popsicles, so the answer is **(D) 13**.

See Also

2017 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by First Problem	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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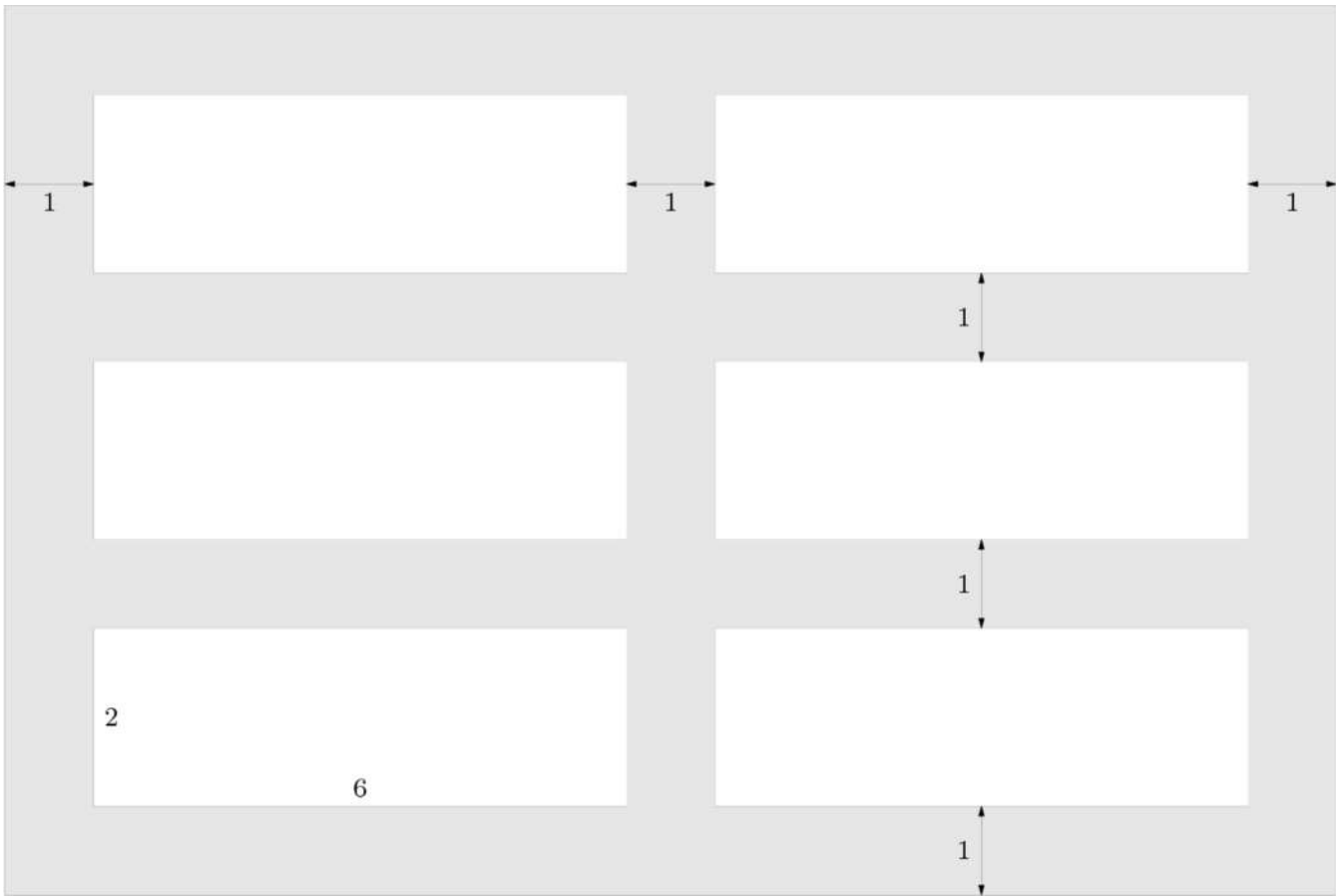


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2017 AMC 10A Problems/Problem 3

Problem

Tamara has three rows of two **6**-feet by **2**-feet flower beds in her garden. The beds are separated and also surrounded by **1**-foot-wide walkways, as shown on the diagram. What is the total area of the walkways, in square feet?



- (A) 72 (B) 78 (C) 90 (D) 120 (E) 150

Solution

Finding the area of the shaded walkway can be achieved by computing the total area of Tamara’s garden and then subtracting the combined area of her six flower beds.

Since the width of Tamara’s garden contains three margins, the total width is $2 \cdot 6 + 3 \cdot 1 = 15$ feet.

Similarly, the height of Tamara’s garden is $3 \cdot 2 + 4 \cdot 1 = 10$ feet.

Therefore, the total area of the garden is $15 \cdot 10 = 150$ square feet.

Finally, since the six flower beds each have an area of $2 \cdot 6 = 12$ square feet, the area we seek is $150 - 6 \cdot 12$, and our answer is **(B) 78**

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 10A Problems/Problem 4

Problem

Mia is “helping” her mom pick up **30** toys that are strewn on the floor. Mia’s mom manages to put **3** toys into the toy box every **30** seconds, but each time immediately after those **30** seconds have elapsed, Mia takes **2** toys out of the box. How much time, in minutes, will it take Mia and her mom to put all **30** toys into the box for the first time?

- (A) 13.5 (B) 14 (C) 14.5 (D) 15 (E) 15.5

Solution

Every **30** seconds $3 - 2 = 1$ toys are put in the box, so after $27 \cdot 30$ seconds there will be **27** toys in the box. Mia’s mom will then put **3** toys into to the box and we have our total amount of time to be $27 \cdot 30 + 30 = 840$ seconds, which equals **14** minutes. **(B) 14**

See also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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2017 AMC 10A Problems/Problem 5

Problem

The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers?

(A) 1 (B) 2 (C) 4 (D) 8 (E) 12

Solution

Let the two real numbers be x, y . We are given that $x + y = 4xy$, and dividing both sides by xy ,

$$\frac{x}{xy} + \frac{y}{xy} = 4.$$

$$\frac{1}{y} + \frac{1}{x} = \boxed{\text{(C) } 4}.$$

Note: we can easily verify that this is the correct answer; for example, $1/2$ and $1/2$ work, and the sum of their reciprocals is 4.

See Also

2017 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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2017 AMC 10A Problems/Problem 6

Problem

Ms. Carroll promised that anyone who got all the multiple choice questions right on the upcoming exam would receive an A on the exam. Which one of these statements necessarily follows logically?

- (A) If Lewis did not receive an A, then he got all of the multiple choice questions wrong.
- (B) If Lewis did not receive an A, then he got at least one of the multiples choice questions wrong.
- (C) If Lewis got at least one of the multiple choice questions wrong, then he did not receive an A.
- (D) If Lewis received an A, then he got all of the multiple choice questions right.
- (E) If Lewis received an A, then he got at least one of the multiple choice questions right.

Solution

Rewriting the given statement: "if someone got all the multiple choice questions right on the upcoming exam then he or she would receive an A on the exam." If that someone is Lewis the statement becomes: "if Lewis got all the multiple choice questions right, then he got an A on the exam." The contrapositive: "If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong (did not get all of them right)" must also be true leaving B as the correct answer. B is also equivalent to the contrapositive of the original statement, which implies that it must be true, so the answer is

(B) If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.

See Also

2017 AMC 10A (Problems • Answer Key • Resources) (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017)	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 12A (Problems • Answer Key • Resources) (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017)	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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2017 AMC 10A Problems/Problem 7

Problem

Jerry and Silvia wanted to go from the southwest corner of a square field to the northeast corner. Jerry walked due east and then due north to reach the goal, but Silvia headed northeast and reached the goal walking in a straight line. Which of the following is closest to how much shorter Silvia's trip was, compared to Jerry's trip?

- (A) 30% (B) 40% (C) 50% (D) 60% (E) 70%

Solution

Let j represent how far Jerry walked, and s represent how far Sylvia walked. Since the field is a square, and Jerry walked two sides of it, while Silvia walked the diagonal, we can simply define the side of the square field to be one, and find the distances they walked. Since Jerry walked two sides, $j = 2$ Since Silvia walked the diagonal, she walked the hypotenuse of a 45, 45, 90 triangle with leg length 1. Thus, $s = \sqrt{2} = 1.414...$ We can then take $\frac{j-s}{j} \approx \frac{2-1.4}{2} = 0.3 \implies \boxed{\text{(A) } 30\%}$

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 12A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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2017 AMC 10A Problems/Problem 8

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See Also

Problem

At a gathering of **30** people, there are **20** people who all know each other and **10** people who know no one. People who know each other a hug, and people who do not know each other shake hands. How many handshakes occur?

(A) 240 (B) 245 (C) 290 (D) 480 (E) 490

Solution 1

Each one of the ten people has to shake hands with all the **20** other people they don't know. So $10 \cdot 20 = 200$. From there, we calculate how many handshakes occurred between the people who don't know each other. This is simply

counting how many ways to choose two people to shake hands, or $\binom{10}{2} = 45$. Thus the answer is

$$200 + 45 = \boxed{\text{(B) } 245}.$$

Solution 2

We can also use complementary counting. First of all, $\binom{30}{2} = 435$ handshakes or hugs occur. Then, if we can find the number of hugs, then we can subtract it from **435** to find the handshakes. Hugs only happen between the 20 people who know each other, so there are $\binom{20}{2} = 190$ hugs. $435 - 190 = \boxed{\text{(B) } 245}$.

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	
2017 AMC 12A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2017 AMC 10A Problems/Problem 9

Problem

Minnie rides on a flat road at **20** kilometers per hour (kph), downhill at **30** kph, and uphill at **5** kph. Penny rides on a flat road at **30** kph, downhill at **40** kph, and uphill at **10** kph. Minnie goes from town **A** to town **B**, a distance of **10** km all uphill, then from town **B** to town **C**, a distance of **15** km all downhill, and then back to town **A**, a distance of **20** km on the flat. Penny goes the other way around using the same route. How many more minutes does it take Minnie to complete the **45**-km ride than it takes Penny?

(A) 45 (B) 60 (C) 65 (D) 90 (E) 95

Solution

The distance from town **A** to town **B** is **10** km uphill, and since Minnie rides uphill at a speed of **5** kph, it will take her **2** hours. Next, she will ride from town **B** to town **C**, a distance of **15** km all downhill. Since Minnie rides downhill at a speed of **30** kph, it will take her half an hour. Finally, she rides from town **C** back to town **A**, a flat distance of **20** km. Minnie rides on a flat road at **20** kph, so this will take her **1** hour. Her entire trip takes her **3.5** hours. Secondly, Penny will go from town **A** to town **C**, a flat distance of **20** km. Since penny rides on a flat road at **30** kph, it will take her $\frac{2}{3}$ of an hour. Next Penny will go from town **C** to town **B**, which is uphill for Penny. Since penny rides at a speed of **10** kph uphill, and town **C** and **B** are **15** km apart, it will take her **1.5** hours. Finally, Penny goes from Town **B** back to town **A**, a distance of **10** km downhill. Since Penny rides downhill at **40** kph, it will only take her $\frac{1}{4}$ of an hour. In total, it takes her $\frac{29}{12}$ hours, which simplifies to **2** hours and **25** minutes. Finally, Penny's **2** Hour **25** Minute trip was **(C) 65** minutes less than Minnie's **3** Hour **30** Minute Trip

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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2017 AMC 10A Problems/Problem 10

Problem

Joy has **30** thin rods, one each of every integer length from **1** cm through **30** cm. She places the rods with lengths **3** cm, **7** cm, and **15** cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

(A) 16 (B) 17 (C) 18 (D) 19 (E) 20

Solution

The triangle inequality generalizes to all polygons, so $x < 3 + 7 + 15$ and $x + 3 + 7 > 15$ to get $5 < x < 25$. Now, we know that there are **19** numbers between **5** and **25** exclusive, but we must subtract **2** to account for the 2 lengths already used that are between those numbers, which gives $19 - 2 = \boxed{\text{(B) } 17}$

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2017 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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2017 AMC 10A Problems/Problem 11

Contents

- 1 Problem
- 2 Solution
- 3 Diagram
- 4 See Also

Problem

The region consisting of all point in three-dimensional space within 3 units of line segment $\overline{AB}$ has volume 216π . What is the length AB ?

(A) 6 (B) 12 (C) 18 (D) 20 (E) 24

Solution

In order to solve this problem, we must first visualize what the region contained looks like. We know that, in a three dimensional plane, the region consisting of all points within **3** units of a point would be a sphere with radius **3**. However, we need to find the region containing all points within 3 units of a segment. It can be seen that our region is a cylinder with two hemispheres on either end. We know the volume of our region, so we set up the following equation (the volume of our cylinder + the volume of our two hemispheres will equal **216π**):

$$\frac{4\pi}{3} \cdot 3^3 + 9\pi x = 216\pi, \text{ where } x \text{ is equal to the length of our line segment.}$$

Solving, we find that $x = \boxed{\text{(D) } 20}$.

Diagram

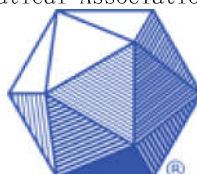
<http://i.imgur.com/cwNt293.png>

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	
2017 AMC 12A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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2017 AMC 10A Problems/Problem 12

Problem

Let S be a set of points (x, y) in the coordinate plane such that two of the three quantities 3 , $x + 2$, and $y - 4$ are equal and the third of the three quantities is no greater than this common value. Which of the following is a correct description for S ?

- (A) a single point (B) two intersecting lines
 (C) three lines whose pairwise intersections are three distinct points
 (D) a triangle (E) three rays with a common endpoint

Solution

If the two equal values are 3 and $x + 2$, then $x = 1$. Also, $y - 4 < 3$ because 3 is the common value. Solving for y , we get $y < 7$. Therefore the portion of the line $x = 1$ where $y < 7$ is part of S . This is a ray with an endpoint of $(1, 7)$.

Similar to the process above, we assume that the two equal values are 3 and $y - 4$. Solving the equation $3 = y - 4$ then $y = 7$. Also, $x + 2 < 3$ because 3 is the common value. Solving for x , we get $x < 1$. Therefore the portion of the line $y = 7$ where $x < 1$ is also part of S . This is another ray with the same endpoint as the above ray: $(1, 7)$.

If $x + 2$ and $y - 4$ are the two equal values, then $x + 2 = y - 4$. Solving the equation for y , we get $y = x + 6$. Also $3 < y - 4$ because $y - 4$ is one way to express the common value. Solving for y , we get $y > 7$. Therefore the portion of the line $y = x + 6$ where $y > 7$ is part of S like the other two rays. The lowest possible value that can be achieved is also $(1, 7)$.

Since S is made up of three rays with common endpoint $(1, 7)$, the answer is

(E) three rays with a common endpoint

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 12A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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2017 AMC 10A Problems/Problem 13

Problem

Define a sequence recursively by $F_0 = 0$, $F_1 = 1$, and $F_n =$ the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geq 2$. Thus the sequence starts 0, 1, 1, 2, 0, 2, ... What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Solution

A pattern starts to emerge as the function is continued. The repeating pattern is 0, 1, 1, 2, 0, 2, 2, 1 ... The problem asks for the sum of eight consecutive terms in the sequence. Because there are eight numbers in the repeating sequence, we just need to find the sum of the numbers in the sequence, which is (D) 9

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017)	
Preceded by Problem 12	Followed by Problem 14
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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2017 AMC 10A Problems/Problem 14

Problem

Every week Roger pays for a movie ticket and a soda out of his allowance. Last week, Roger's allowance was A dollars. The cost of his movie ticket was **20%** of the difference between A and the cost of his soda, while the cost of his soda was **5%** of the difference between A and the cost of his movie ticket. To the nearest whole percent, what fraction of A did Roger pay for his movie ticket and soda?

(A) 9% (B) 19% (C) 22% (D) 23% (E) 25%

Solution

Let m = cost of movie ticket
Let s = cost of soda

We can create two equations:

$$m = \frac{1}{5}(A - s)$$

$$s = \frac{1}{20}(A - m)$$

Substituting we get:

$$m = \frac{1}{5}\left(A - \frac{1}{20}(A - m)\right)$$

which yields:

$$m = \frac{19}{99}A$$

Now we can find s and we get:

$$s = \frac{4}{99}A$$

Since we want to find what fraction of A did Roger pay for his movie ticket and soda, we add m and s to get:

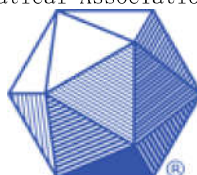
$$\frac{19}{99}A + \frac{4}{99}A \Rightarrow \boxed{\text{(D) } 23\%}$$

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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2017 AMC 10A Problems/Problem 15

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See Also

Problem

Chlo   chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chlo  's number?

- (A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{5}{6}$ (E) $\frac{7}{8}$

Solution 1

Denote "winning" to mean "picking a greater number". There is a $\frac{1}{2}$ chance that Laurent chooses a number in the interval $[2017, 4034]$. In this case, Chlo   cannot possibly win, since the maximum number she can pick is 2017.

Otherwise, if Laurent picks a number in the interval $[0, 2017]$, with probability $\frac{1}{2}$, then the two people are

symmetric, and each has a $\frac{1}{2}$ chance of winning. Then, the total probability is $\frac{1}{2} * 1 + \frac{1}{2} * \frac{1}{2} = \boxed{\text{(C)} \frac{3}{4}}$

Solution 2

We can use geometric probability to solve this. Suppose a point (x, y) lies in the xy -plane. Let x be Chlo  's number and y be Laurent's number. Then obviously we want $y > x$, which basically gives us a region above a line. We know that Chlo  's number is in the interval $[0, 2017]$ and Laurent's number is in the interval $[0, 4034]$, so we can create a rectangle in the plane, whose length is 2017 and whose width is 4034. Drawing it out, we see that it is easier to find the probability that Chlo  's number is greater than Laurent's number and subtract this probability from 1. The probability that Chlo  's number is larger than Laurent's number is simply the area of the

region under the line $y > x$, which is $\frac{2017 \cdot 2017}{2}$. Instead of bashing this out we know that the rectangle has

area $2017 \cdot 4034$. So the probability that Laurent has a smaller number is $\frac{2017 \cdot 2017}{2 \cdot 2017 \cdot 4034}$. Simplifying the

expression yields $\frac{1}{4}$ and so $1 - \frac{1}{4} = \boxed{\text{(C)} \frac{3}{4}}$.

~AoPS12142015

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 10A Problems/Problem 16

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See Also

Problem

There are 10 horses, named Horse 1 , Horse 2 , . . . , Horse 10 . They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds. The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of T ?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution 1

If we have horses, $a_1, a_2, \dots, a_n$, then any number that is a multiple of the all those numbers is a time when all horses will meet at the starting point. The least of these numbers is the LCM. To minimize the LCM, we need the smallest primes, and we need to repeat them a lot. By inspection, we find that $\text{LCM}(1, 2, 3, 2 \cdot 2, 2 \cdot 3) = 12$. Finally, $1 + 2 = \boxed{\text{(B)} 3}$.

Solution 2

We are trying to find the smallest number that has 5 one-digit divisors. We quickly consider 12 since it is the smallest number that has at least 5 divisors. Since 12 has 5 single-digit divisors, namely $1, 2, 3, 4$, and 6 , our answer is $1 + 2 = \boxed{\text{(B)} 3}$.

See Also

2017 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 15	Followed by Problem 17
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2017 AMC 10A Problems/Problem 17

Problem

Distinct points P , Q , R , S lie on the circle $x^2 + y^2 = 25$ and have integer coordinates. The distances PQ and RS are irrational numbers. What is the greatest possible value of the ratio $\frac{PQ}{RS}$?

- (A) 3 (B) 5 (C) $3\sqrt{5}$ (D) 7 (E) $5\sqrt{2}$

Solution

Because P , Q , R , and S are integers there are only a few coordinates that actually satisfy the equation. The coordinates are $(\pm 3, \pm 4)$, $(\pm 4, \pm 3)$, $(0, \pm 5)$, and $(\pm 5, 0)$. We want to maximize PQ and minimize RS . They also have to be the square root of something, because they are both irrational. The greatest value of PQ happens when it P and Q are almost directly across from each other and are in different quadrants. For example, the endpoints of the segment could be $(4, 3)$ and $(-3, -4)$ because the two points are almost across from each other. The least value of RS is when the two endpoints are in the same quadrant and are very close to each other. This can occur when, for example, R is $(3, 4)$ and S is $(4, 3)$. They are in the same quadrant and no other point on the circle with integer coordinates is closer to the point $(3, 4)$ than $(4, 3)$. Using the distance formula, we get that

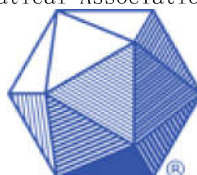
$$PQ \text{ is } \sqrt{98} \text{ and that } RS \text{ is } \sqrt{2}. \frac{\sqrt{98}}{\sqrt{2}} = \sqrt{49} = \boxed{\text{(D) } 7}$$

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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2017 AMC 10A Problems/Problem 18

Contents

▪ 1 Problem

▪ 2 Solution

▪ 3 Solution 2

▪ 4 See Also

Problem

Amelia has a coin that lands heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone gets a head; the first one to get a head wins. All coin tosses are independent. Amelia goes first. The probability that Amelia wins is $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $q - p$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution

Let P be the probability Amelia wins. Note that $P = \text{chance she wins on her first turn} + \text{chance she gets to her turn again} \cdot P$, as if she gets to her turn again, she is back where she started with probability of winning P . The chance she wins on her first turn is $\frac{1}{3}$. The chance she makes it to her turn again is a combination of her failing to win the first turn - $\frac{2}{3}$ and Blaine failing to win - $\frac{3}{5}$. Multiplying gives us $\frac{2}{5}$. Thus,

$$P = \frac{1}{3} + \frac{2}{5}P$$

Therefore, $P = \frac{5}{9}$, so the answer is $9 - 5 = \boxed{\text{(D)}\ 4}$.

Solution 2

Let P be the probability Amelia wins. Note that $P = \text{chance she wins on her first turn} + \text{chance she gets to her second turn} \cdot \frac{1}{3} + \text{chance she gets to her third turn} \cdot \frac{1}{3}..$ This can be represented by an infinite geometric series,

$$P = \frac{\frac{1}{3}}{1 - \frac{2}{3} \cdot \frac{3}{5}} = \frac{\frac{1}{3}}{1 - \frac{2}{5}} = \frac{\frac{1}{3}}{\frac{3}{5}} = \frac{1}{3} \cdot \frac{5}{3} = \frac{5}{9}$$

. Therefore, $P = \frac{5}{9}$, so the answer is $9 - 5 = \boxed{\text{(D)}\ 4}$

Solution by ktong

See Also

2017 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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2017 AMC 10A Problems/Problem 19

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See Also

Problem

Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

(A) 12 (B) 16 (C) 28 (D) 32 (E) 40

Solution 1

For notation purposes, let Alice be A, Bob be B, Carla be C, Derek be D, and Eric be E. We can split this problem up into two cases:

Case 1: A sits on an edge seat.

Then, since B and C can't sit next to A, that must mean either D or E sits next to A. After we pick either D or E, then either B or C must sit next to D/E. Then, we can arrange the two remaining people in two ways. Since there are two different edge seats that A can sit in, there are a total of $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

Case 2: A does not sit in an edge seat.

In this case, then only two people that can sit next to A are D and E, and there are two ways to permute them, and this also handles the restriction that D can't sit next to E. Then, there are two ways to arrange B and C, the remaining people. However, there are three initial seats that A can sit in, so there are $3 \cdot 2 \cdot 2 = 12$ seatings in this case.

Adding up all the cases, we have $16 + 12 = \boxed{\text{(C) } 28}$.

Solution 2

Label the seats **1** through **5**. The number of ways to seat Derek and Eric in the five seats with no restrictions is $5 \cdot 4 = 20$. The number of ways to seat Derek and Eric such that they sit next to each other is **8** (which can be figure out quickly), so the number of ways such that Derek and Eric don't sit next to each other is $20 - 8 = 12$. Note that once Derek and Eric are seated, there are three cases.

The first case is that they sit at each end. There are two ways to seat Derek and Eric. But this is impossible because then Alice, Bob, and Carla would have to sit in some order in the middle three seats which would lead to Alice sitting next to Bob or Carla, a contradiction. So this case gives us **0** ways.

Another possible case is if Derek and Eric seat in seats **2** and **4** in some order. There are 2 possible ways to seat Derek and Eric like this. This leaves Alice, Bob, and Carla to sit in any order in the remaining three seats. Since no two of these three seats are consecutive, there are $3! = 6$ ways to do this. So the second case gives us $2 \cdot 6 = 12$ total ways for the second case.

The last case is if once Derek and Eric are seated, exactly one pair of consecutive seats is available. There are $12 - 2 - 2 = 8$ ways to seat Derek and Eric like this. Once they are seated like this, Alice cannot sit in one of the two consecutive available seats without sitting next to Bob and Carla. So Alice has to sit in the other remaining chair. Then, there are two ways to seat Bob and Carla in the remaining two seats (which are consecutive). So this case gives us $8 \cdot 2 = 16$ ways.

So in total there are $12 + 16 = 28$. So our answer is $\boxed{\text{(C) } 28}$.

See Also

2017 AMC 10A Problems/Problem 20

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See Also

Problem

Let $S(n)$ equal the sum of the digits of positive integer n . For example, $S(1507) = 13$. For a particular positive integer n , $S(n) = 1274$. Which of the following could be the value of $S(n+1)$?

(A) 1 (B) 3 (C) 12 (D) 1239 (E) 1265

Solution 1

Note that $n \equiv S(n) \pmod{9}$. This can be seen from the fact that $\sum_{k=0}^n 10^k a_k \equiv \sum_{k=0}^n a_k \pmod{9}$. Thus, if $S(n) = 1274$, then $n \equiv 5 \pmod{9}$, and thus $n+1 \equiv S(n+1) \equiv 6 \pmod{9}$. The only answer choice that is $6 \pmod{9}$ is **(D) 1239**.

Solution 2

We can find out that the least number of digits the number N is 142, with 141 9's and one 5. By randomly mixing the digits up, we are likely to get: 9999...9995999...9999. By adding 1 to this number, we get: 9999...9996000...0000. Knowing that this number is ONLY divisible by 9 when 6 is subtracted, we can subtract 6 from every available choice, and see if the number is divisible by 9 afterwards. After subtracting 6 from every number, we can conclude that 1233 (originally 1239) is the only number divisible by 9. So our answer is **(D) 1239**.

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	
2017 AMC 12A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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2017 AMC 10A Problems/Problem 21

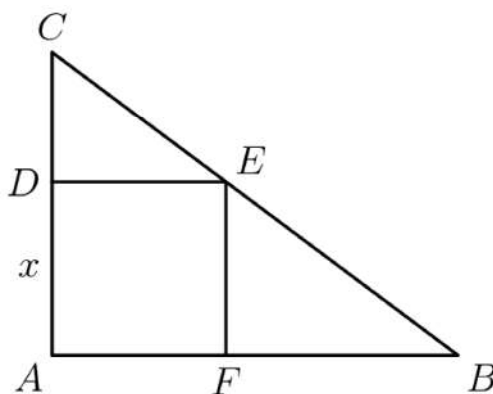
Problem

A square with side length x is inscribed in a right triangle with sides of length **3**, **4**, and **5** so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length **3**, **4**, and **5** so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

- (A) $\frac{12}{13}$ (B) $\frac{35}{37}$ (C) 1 (D) $\frac{37}{35}$ (E) $\frac{13}{12}$

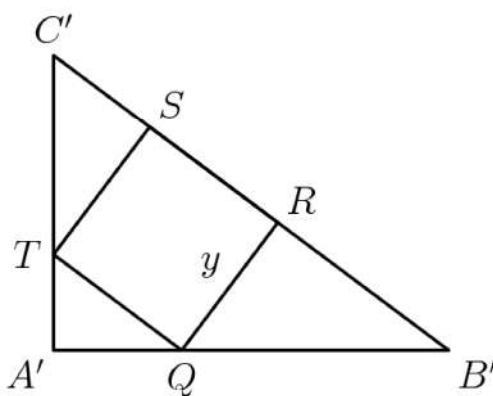
Solution

Analyze the first right triangle.



Note that $\triangle ABC$ and $\triangle FBE$ are similar, so $\frac{BF}{FE} = \frac{AB}{AC}$. This can be written as $\frac{4-x}{x} = \frac{4}{3}$. Solving, $x = \frac{12}{7}$.

Now we analyze the second triangle.



Similarly, $\triangle A'B'C'$ and $\triangle RB'Q$ are similar, so $RB' = \frac{4}{3}y$, and $C'S = \frac{3}{4}y$. Thus, $C'B' = C'S + SR + RB' = \frac{4}{3}y + y + \frac{3}{4}y = 5$. Solving for y , we get $y = \frac{60}{37}$. Thus,

$$\frac{x}{y} = \boxed{\text{(D)} \frac{37}{35}}.$$

See Also

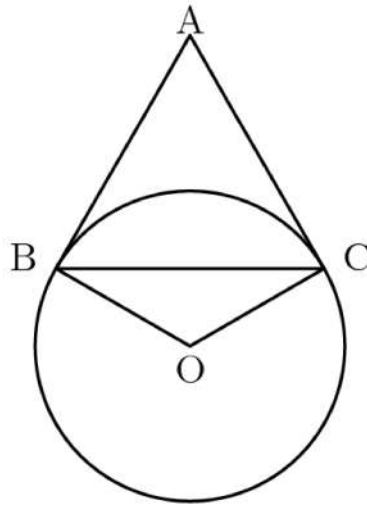
2017 AMC 10A Problems/Problem 22

Problem

Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

- (A) $\frac{4\sqrt{3}\pi}{27} - \frac{1}{3}$ (B) $\frac{\sqrt{3}}{2} - \frac{\pi}{8}$ (C) $\frac{1}{2}$ (D) $\sqrt{3} - \frac{2\sqrt{3}\pi}{9}$ (E) $\frac{4}{3} - \frac{4\sqrt{3}\pi}{27}$

Solution



Let the radius of the circle be r , and let its center be O . Since $\overline{AB}$ and $\overline{AC}$ are tangent to circle O , then $\angle OBA = \angle OCA = 90^\circ$, so $\angle BOC = 120^\circ$. Therefore, since $\overline{OB}$ and $\overline{OC}$ are equal to r , then (pick your favorite method) $\overline{BC} = r\sqrt{3}$. The area of the equilateral triangle is $\frac{(r\sqrt{3})^2\sqrt{3}}{4} = \frac{3r^2\sqrt{3}}{4}$, and the area of the sector we are subtracting from it is $\frac{1}{3}\pi r^2 - \frac{1}{2}r \cdot r \cdot \frac{\sqrt{3}}{2} = \frac{\pi r^2}{3} - \frac{r^2\sqrt{3}}{4}$. The area outside of the triangle is $\frac{3r^2\sqrt{3}}{4} - \left(\frac{\pi r^2}{3} - \frac{r^2\sqrt{3}}{4}\right) = r^2\sqrt{3} - \frac{\pi r^2}{3}$. Therefore, the answer is

$$\frac{r^2\sqrt{3} - \frac{\pi r^2}{3}}{\frac{3r^2\sqrt{3}}{4}} = \boxed{\text{(E)} \frac{4}{3} - \frac{4\sqrt{3}\pi}{27}}$$

See Also

2017 AMC 10A (Problems • Answer Key • Resources)	
(http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 21	Followed by Problem 23
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 10A Problems/Problem 23

Problem

How many triangles with positive area have all their vertices at points (i, j) in the coordinate plane, where i and j are integers between **1** and **5**, inclusive?

- (A)** 2128 **(B)** 2148 **(C)** 2160 **(D)** 2200 **(E)** 2300

Solution

There are a total of $\binom{25}{3} = 2300$ sets of three points. However, some of them form degenerate triangles (i.e., they have area of 0) if the three points are collinear. There are a total of 12 lines that go through 5 points (5 vertical, 5 horizontal, 2 diagonal), which contributes $\binom{5}{3} \cdot 12 = 120$ degenerate triangles, 4 lines that go through exactly 4 points, which contributes $\binom{4}{3} \cdot 4 = 16$ degenerate triangles, and 16 lines that go through exactly three points, which contributes $\binom{3}{3} \cdot 16 = 16$ degenerate triangles. Subtracting these degenerate triangles, we get an answer of $2300 - 120 - 16 - 16 = 2300 - 152 = \boxed{\text{(B) } 2148}$.

See Also

2017 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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2017 AMC 10A Problems/Problem 24

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4 (Slight guessing)
- 6 Solution 5
- 7 See Also

Problem

For certain real numbers a , b , and c , the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

- (A) -9009 (B) -8008 (C) -7007 (D) -6006 (E) -5005

Solution 1

$f(x)$ must have four roots, three of which are roots of $g(x)$. Using the fact that every polynomial has a unique factorization into its roots, and since the leading coefficient of $f(x)$ and $g(x)$ are the same, we know that

$$f(x) = g(x)(x - r)$$

where $r \in \mathbb{C}$ is the fourth root of $f(x)$. Substituting $g(x)$ and expanding, we find that

$$\begin{aligned} f(x) &= (x^3 + ax^2 + x + 10)(x - r) \\ &= x^4 + (a - r)x^3 + (1 - ar)x^2 + (10 - r)x - 10r. \end{aligned}$$

Comparing coefficients with $f(x)$, we see that

$$\begin{aligned} a - r &= 1 \\ 1 - ar &= b \\ 10 - r &= 100 \\ -10r &= c. \end{aligned}$$

Let's solve for a , b , c , and r . Since $10 - r = 100$, $r = -90$, so $c = (-10)(-90) = 900$. Since $a - r = 1$, $a = -89$, and $b = 1 - ar = -8009$. Thus, we know that

$$f(x) = x^4 + x^3 - 8009x^2 + 100x + 900.$$

Taking $f(1)$, we find that

$$\begin{aligned}
 f(1) &= 1^4 + 1^3 - 8009(1)^2 + 100(1) + 900 \\
 &= 1 + 1 - 8009 + 100 + 900 \\
 &= \boxed{(\mathbf{C}) - 7007}.
 \end{aligned}$$

Solution 2

We notice that the constant term of $f(x) = c$ and the constant term in $g(x) = 10$. Because $f(x)$ can be factored as $g(x) \cdot (x - r)$ (where r is the unshared root of $f(x)$), we see that using the constant term, $-10 \cdot r = c$ and therefore $r = -\frac{c}{10}$. Now we once again write $f(x)$ out in factored form:

$$f(x) = g(x) \cdot (x - r) = (x^3 + ax^2 + x + 10)(x + \frac{c}{10})$$

We can expand the expression on the right-hand side to get:

$$f(x) = x^4 + (a + \frac{c}{10})x^3 + (1 + \frac{ac}{10})x^2 + (10 + \frac{c}{10})x + c$$

Now we have

$$f(x) = x^4 + (a + \frac{c}{10})x^3 + (1 + \frac{ac}{10})x^2 + (10 + \frac{c}{10})x + c = x^4 + x^3 + bx^2 + 100x + c.$$

Simply looking at the coefficients for each corresponding term (knowing that they must be equal), we have the equations:

$$10 + \frac{c}{10} = 100 \Rightarrow c = 900$$

$$a + \frac{c}{10} = 1, c = 900 \Rightarrow a + 90 = 1 \Rightarrow a = -89$$

and finally,

$$1 + \frac{ac}{10} = b = 1 + \frac{-89 \cdot 900}{10} = b = -8009$$

We know that $f(1)$ is the sum of its coefficients, hence $1 + 1 + b + 100 + c$. We substitute the values we obtained for b and c into this expression to get

$$f(1) = 1 + 1 + (-8009) + 100 + 900 = \boxed{(\mathbf{C}) - 7007}.$$

Solution 3

Let r_1, r_2 , and r_3 be the roots of $g(x)$. Let r_4 be the additional root of $f(x)$. Then from Vieta's formulas on the quadratic term of $g(x)$ and the cubic term of $f(x)$, we obtain the following:

$$\begin{aligned}
 r_1 + r_2 + r_3 &= -a \\
 r_1 + r_2 + r_3 + r_4 &= -1
 \end{aligned}$$

Thus $r_4 = a - 1$.

Now applying Vieta's formulas on the constant term of $g(x)$, the linear term of $g(x)$, and the linear term of $f(x)$, we obtain:

$$\begin{aligned}
 r_1 r_2 r_3 &= -10 \\
 r_1 r_2 + r_2 r_3 + r_3 r_1 &= 1 \\
 r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2 &= -100
 \end{aligned}$$

Substituting for $r_1 r_2 r_3$ in the bottom equation and factoring the remainder of the expression, we obtain:

$$-10 + (r_1 r_2 + r_2 r_3 + r_3 r_1) r_4 = -10 + r_4 = -100$$

It follows that $r_4 = -90$. But $r_4 = a - 1$ so $a = -89$

Now we can factor $f(x)$ in terms of $g(x)$ as

$$f(x) = (x - r_4)g(x) = (x + 90)g(x)$$

Then $f(1) = 91g(1)$ and

$$g(1) = 1^3 - 89 \cdot 1^2 + 1 + 10 = -77$$

Hence $f(1) = 91 \cdot (-77) = \boxed{(C) - 7007}$.

Solution 4 (Slight guessing)

Let the roots of $g(x)$ be r_1 , r_2 , and r_3 . Let the roots of $f(x)$ be r_1 , r_2 , r_3 , and r_4 . From Vieta's, we have:

$$\begin{aligned}
 r_1 + r_2 + r_3 &= -a \\
 r_1 + r_2 + r_3 + r_4 &= -1 \\
 r_4 &= a - 1
 \end{aligned}$$

The fourth root is $a - 1$. Since r_1 , r_2 , and r_3 are common roots, we have:

$$\begin{aligned}
 f(x) &= g(x)(x - (a - 1)) \\
 f(1) &= g(1)(1 - (a - 1)) \\
 f(1) &= (a + 12)(2 - a) \\
 f(1) &= -(a + 12)(a - 2)
 \end{aligned}$$

Let $a - 2 = k$:

$$f(1) = -k(k + 14)$$

Note that $-7007 = -1001 \cdot (7) = -(7 \cdot (11) \cdot (13)) \cdot (7) = -91 \cdot (77)$ This gives us a pretty good guess of $\boxed{(C) - 7007}$.

Solution 5

First off, let's get rid of the x^4 term by finding $h(x) = f(x) - xg(x)$. This polynomial consists of the difference of two polynomials with 3 common factors, so it must also have these factors. The polynomial is $h(x) = (1 - a)x^3 + (b - 1)x^2 + 90x + c$, and must be equal to $(1 - a)g(x)$. Equating the coefficients, we get 4 equations. We will tackle them one at a time, starting the x terms.

$$90 = 1 - a$$

The solution is obviously $a = -89$. We can now find b and c with ease.

$$b - 1 = a(1 - a) = -89 * 90 = -8010$$

and $b = -8009$.

$$c = 10(1 - a) = 10 * 90 = 900$$

Finally, $f(1) = 102 + b + c = 102 + 900 - 8009 = \boxed{(C) - 7007}.$

See Also

2017 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
Preceded by Problem 23	Followed by Problem 25
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2017 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2017))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

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2017 AMC 10A Problems/Problem 25

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3 (Shorter and Not Casework)
- 5 See Also

Problem

How many integers between **100** and **999**, inclusive, have the property that some permutation of its digits is a multiple of **11** between **100** and **999**? For example, both **121** and **211** have this property.

(A) 226 (B) 243 (C) 270 (D) 469 (E) 486

Solution 1

Let the three-digit number be ACB :

If a number is divisible by **11**, then the difference between the sums of alternating digits is a multiple of **11**.

There are two cases: $A + B = C$ and $A + B = C + 11$

We now proceed to break down the cases.

Case 1: $A + B = C$.

Part 1: $B = 0, A = C$, this case results in 110, 220, 330...990. There are two ways to arrange the digits in each of those numbers. $2 \cdot 9 = 18$

Part 2: $B > 0, B = 1, A + 1 = C$, this case results in 121, 231,... 891. There are **6** ways to arrange the digits in all of those number except the first, and 3 ways ways for the first. This leads to **45** cases.

Part 3: $B = 2, A + 2 = C$, this case results in 242, 352,... 792. There are **6** ways to arrange the digits in all of those number except the first, and 3 ways ways for the first. This leads to **33** cases.

Part 4: $B = 3, A + 3 = C$, this case results in 363, 473,...693. There are **6** ways to arrange the digits in all of those number except the first, and 3 ways ways for the first. This leads to **21** cases.

Part 5: $B = 4, A + 4 = C$, this case results in 484 and 594. There are **6** ways to arrange the digits in all of those number except the first, and 3 ways ways for the first. This leads to **9** cases.

This case has $18 + 45 + 33 + 21 + 9 = 126$ subcases.

Case 2: $A + B = C + 11$.

Part 1: $C = 0, A + B = 11$, this cases results in 209, 308, ...506. There are **4** ways to arrange each of those cases. This leads to **16** cases.

Part 2: $C = 1, A + B = 12$, this cases results in 319, 418, ...616. There are **6** ways to arrange each of those cases, except the last. This leads to **21** cases.

Part 3: $C = 2, A + B = 13$, this cases results in 429, 528, ...617. There are **6** ways to arrange each of those cases. This leads to **18** cases.

... If we continue this counting, we receive $16 + 21 + 18 + 15 + 12 + 9 + 6 + 3 = 100$ subcases.

$$100 + 126 = \boxed{\text{(A) } 226}$$

~Mathguy1492

Solution 2

We note that we only have to consider multiples of 11 and see how many valid permutations each has. We can do casework on the number of repeating digits that the multiple of 11 has:

Case 1: All three digits are the same. By inspection, we find that there are no multiples of 11 here.

Case 2: Two of the digits are the same, and the third is different.

Case 2a: There are 8 multiples of 11 without a zero that have this property: 121, 242, 363, 484, 616, 737, 858, 979. Each contributes 3 valid permutations, so there are $8 \cdot 3 = 24$ permutations in this subcase.

Case 2b: There are 9 multiples of 11 with a zero that have this property: 110, 220, 330, 440, 550, 660, 770, 880, 990. Each one contributes 2 valid permutations (the first digit can't be zero), so there are $9 \cdot 2 = 18$ permutations in this subcase.

Case 3: All the digits are different. Since there are $\frac{990 - 110}{11} + 1 = 81$ multiples of 11 between 100 and 999, there are $81 - 8 - 9 = 64$ multiples of 11 remaining in this case. However, 8 of them contain a zero, namely 209, 308, 407, 506, 605, 704, 803, and 902. Each of those multiples of 11 contributes $2 \cdot 2 = 4$ valid permutations, but we overcounted by a factor of 2; every permutation of 209, for example, is also a permutation of 902. Therefore, there are $8 \cdot 4/2 = 16$. Therefore, there are $64 - 8 = 56$ remaining multiples of 11 without a 0 in this case. Each one contributes $3! = 6$ valid permutations, but once again, we overcounted by a factor of 2 (note that if a number ABC is a multiple of 11, then so is CBA). Therefore, there are $56 \cdot 6/2 = 168$ valid permutations in this subcase.

Adding up all the permutations from all the cases, we have $24 + 18 + 16 + 168 = \boxed{(A) 226}$.

Solution 3 (Shorter and Not Casework)

We can overcount and then subtract. We know there are 81 multiples of 11.

We can multiply by 6 for each permutation of these multiples. (Yet some multiples don't have 6)

Now divide by 2, because if a number abc with digits a , b , and c is a multiple of 11, then cab is also a multiple of 11 so we have counted the same permutations twice.

Basically, each multiple of 11 has its own 3 permutations (say abc has abc , acb and bac whereas cab has cab , cba and bca). We know that each multiple of 11 has at least 3 permutations because it cannot have 3 repeating digits.

Hence we have 243 permutations without subtracting for over count. Now note that we overcounted cases in which we have 0's at the start of each number. So in theory we could just answer A and move on.

We can also solve it :/ We overcount cases where the Middle digit of the number is 0 and the last digit is 0.

Note that we assigned each multiple of 11 3 permutations.

The last digit is 0 gives 9 possibilities where we overcount by 1 permutation for each of 110, 220, ..., 990.

The middle digit is 0 gives 8 possibilities where we overcount by 1. 605, 704, 803, 902 and 506, 407, 308, 209

Subtracting 17 gives $\boxed{(A) 226}$.

Now, we may ask if there is further overlap (I.e if two of abc and bac and acb were multiples of 11) Thankfully, using divisibility rules, this can never happen as taking the divisibility rule mod 11 and adding we get that $2a$, $2b$, or $2c$ is congruent to 0 (mod 11). Since a, b, c are digits, this can never happen as none of them can equal 11 and they can't equal 0 as they are the leading digit of a 3 digit number in each of the cases

~fuzz1

See Also