

2017 AMC 12A Problems/Problem 1

Problem

Pablo buys popsicles for his friends. The store sells single popsicles for **\$1** each, 3-popsicle boxes for **\$2**, and 5-popsicle boxes for **\$3**. What is the greatest number of popsicles that Pablo can buy with **\$8**?

- (A) 8 (B) 11 (C) 12 (D) 13 (E) 15

Solution

By the greedy algorithm, we can take two 5-popsicle boxes and one 3-popsicle box with **\$8**. To prove that this is optimal, consider an upper bound as follows: at the rate of **\$3** per 5 popsicles, we can get $\frac{40}{3}$ popsicles, which is less than 14. **D**.

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2017 AMC 12A Problems/Problem 2

Problem

The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers?

- (A) 1 (B) 2 (C) 4 (D) 8 (E) 12

Solution

Let x, y be our two numbers. Then $x + y = 4xy$. Thus,

$$\frac{x + y}{xy} = \frac{1}{x} + \frac{1}{y} = 4.$$

C.

See Also

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2017 AMC 12A Problems/Problem 3

Problem

Ms. Carroll promised that anyone who got all the multiple choice questions right on the upcoming exam would receive an A on the exam. Which one of these statements necessarily follows logically?

- (A) If Lewis did not receive an A, then he got all of the multiple choice questions wrong.
- (B) If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.
- (C) If Lewis got at least one of the multiple choice questions wrong, then he did not receive an A.
- (D) If Lewis received an A, then he got all of the multiple choice questions right.
- (E) If Lewis received an A, then he got at least one of the multiple choice questions right.

Solution

Taking the contrapositive of the statement "if he got all of them right, he got an A" yields "if he didn't get an A, he didn't get all of them right", yielding the answer **(B)**.

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2017 AMC 12A Problems/Problem 4

Problem

Jerry and Silvia wanted to go from the southwest corner of a square field to the northeast corner. Jerry walked due east and then due north to reach the goal, but Silvia headed northeast and reached the goal walking in a straight line. Which of the following is closest to how much shorter Silvia's trip was, compared to Jerry's trip?

- (A) 30% (B) 40% (C) 50% (D) 60% (E) 70%

Solution

Let j represent how far Jerry walked, and s represent how far Sylvia walked. Since the field is a square, and Jerry walked two sides of it, while Silvia walked the diagonal, we can simply define the side of the square field to be one, and find the distances they walked. Since Jerry walked two sides, $j = 2$ Since Silvia walked the diagonal, she walked the hypotenuse of a 45, 45, 90 triangle with leg length 1. Thus, $s = \sqrt{2} = 1.414...$ We can then take $\frac{j-s}{j} \approx \frac{2-1.4}{2} = 0.3 = 30\%$ **A**.

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2017 AMC 12A Problems/Problem 5

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Problem

At a gathering of **30** people, there are **20** people who all know each other and **10** people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

(A) 240 (B) 245 (C) 290 (D) 480 (E) 490

Solution

Let the group of people who all know each other be A , and let the group of people who know no one be B . Handshakes occur between each pair (a, b) such that $a \in A$ and $b \in B$, and between each pair of members in B . Thus, the answer is

$$|A||B| + \binom{|B|}{2} = 20 \cdot 10 + \binom{10}{2} = 200 + 45 = \boxed{(B) = 245}$$

Alternate solution

The number of handshakes will be equivalent to the difference between the number of total interactions and the number of hugs, which are $\binom{30}{2}$ and $\binom{20}{2}$, respectively. Thus the total amount of handshakes is

$$\binom{30}{2} - \binom{20}{2} = 435 - 190 = \boxed{(B) = 245}$$

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2017 AMC 12A Problems/Problem 6

Problem

Joy has **30** thin rods, one each of every integer length from **1 cm** through **30 cm**. She places the rods with lengths **3 cm**, **7 cm**, and **15cm** on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

(A) 16 (B) 17 (C) 18 (D) 19 (E) 20

Solution

The quadrilateral cannot be a straight line. Thus, the fourth side must be longer than $15 - (3 + 7) = 5$ and shorter than $15 + 3 + 7 = 25$. This means Joy can use the 19 possible integer rod lengths that fall into $[6, 24]$. However, she has already used the rods of length **7 cm** and **15 cm** so the answer is $19 - 2 = 17$ **(B)**

See Also

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2017 AMC 12A Problems/Problem 7

Problem

Define a function on the positive integers recursively by $f(1) = 2$, $f(n) = f(n - 1) + 2$ if n is even, and $f(n) = f(n - 2) + 2$ if n is odd and greater than 1. What is $f(2017)$?

(A) 2017 (B) 2018 (C) 4034 (D) 4035 (E) 4036

Solution

This is a recursive function, which means the function is used to evaluate itself. To solve this, we must identify the base case, $f(1) = 2$. We also know that when n is odd, $f(n) = f(n - 2) + 2$. Thus we know that $f(2017) = f(2015) + 2$. Thus we know that n will always be odd in the recursion of $f(2017)$, and we add 2 each recursive cycle, which there are 1008 of. Thus the answer is $1008 \cdot 2 + 2 = 2018$, which is answer

(B)

Alternatively, simply download java and run this method and input $n = 2017$:

```
public static int f(int n){
    if(n == 1)
        return 2;
    else if(n % 2 == 0)
        return f(n-1) + 2;
    else
        return f(n-2) + 2;
}
```

It will return $f(2017) = 2018$ which is answer choice **(B)**

See Also

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2017 AMC 12A Problems/Problem 8

Problem

The region consisting of all points in three-dimensional space within **3** units of line segment $\overline{AB}$ has volume 216π . What is the length AB ?

- (A) 6 (B) 12 (C) 18 (D) 20 (E) 24

Solution

Let the length AB be L . Then, we see that the region is just the union of the cylinder with central axis $\overline{AB}$ and radius **3** and the two hemispheres connected to each face of the cylinder (also with radius **3**). Thus the volume is

$$9\pi L + \frac{4}{3}\pi(3)^3 = 9\pi L + 36\pi (= 216\pi)$$

$$9\pi L = 180\pi$$

$$L = \boxed{(D) = 20}$$

See Also

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2017 AMC 12A Problems/Problem 9

Problem

Let S be the set of points (x, y) in the coordinate plane such that two of the three quantities 3 , $x + 2$, and $y - 4$ are equal and the third of the three quantities is no greater than the common value. Which of the following is a correct description of S ?

- (A) a single point (B) two intersecting lines
 (C) three lines whose pairwise intersections are three distinct points
 (D) a triangle (E) three rays with a common point

Solution

If the two equal values are 3 and $x + 2$, then $x = 1$. Also, $y - 4 < 3$ because 3 is the common value. Solving for y , we get $y < 7$. Therefore the portion of the line $x = 1$ where $y < 7$ is part of S . This is a ray with an endpoint of $(1, 7)$.

Similar to the process above, we assume that the two equal values are 3 and $y - 4$. Solving the equation $3 = y - 4$ then $y = 7$. Also, $x + 2 < 3$ because 3 is the common value. Solving for x , we get $x < 1$. Therefore the portion of the line $y = 7$ where $x < 1$ is also part of S . This is another ray with the same endpoint as the above ray: $(1, 7)$.

If $x + 2$ and $y - 4$ are the two equal values, then $x + 2 = y - 4$. Solving the equation for y , we get $y = x + 6$. Also $3 < y - 4$ because $y - 4$ is one way to express the common value. Solving for y , we get $y > 7$. Therefore the portion of the line $y = x + 6$ where $y > 7$ is part of S like the other two rays. The lowest possible value that can be achieved is also $(1, 7)$.

Since S is made up of three rays with common endpoint $(1, 7)$, the answer is **E**

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2017 AMC 12A Problems/Problem 10

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Problem

Chlo   chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chlo  's number?

- (A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{5}{6}$ (E) $\frac{7}{8}$

Solution

Suppose Laurent's number is in the interval $[0, 2017]$. Then, by symmetry, the probability of Laurent's number being greater is $\frac{1}{2}$. Next, suppose Laurent's number is in the interval $[2017, 4034]$. Then Laurent's number will be greater with probability $\frac{1}{2}$. Since each case is equally likely, the probability of Laurent's number being greater is $\frac{1 + \frac{1}{2}}{2} = \frac{3}{4}$, so the answer is C.

Alternate Solution: Geometric Probability

Let x be the number chosen randomly by Chlo  . Because it is given that the number Chlo   choose is interval $[0, 2017]$, $0 \leq x \leq 2017$. Next, let y be the number chosen randomly by Laurent. Because it is given that the number Chlo   choose is interval $[0, 2017]$, $0 \leq y \leq 4034$. Since we are looking for when Laurent's number is Chlo  's we write the equation $y > x$. When these three inequalities are graphed the area captured by $0 \leq x \leq 2017$ and $0 \leq y \leq 4034$ represents all the possibilities, forming a rectangle 2017 in width and 4034 in height. Thus making its area $4034 \cdot 2017$. The area captured by $0 \leq x \leq 2017$, $0 \leq y \leq 4034$, and $y > x$ represents the possibilities of Laurent winning, forming a trapezoid with a height 2017 in length and bases 4034 and 2017 length, thus making an area $2017 \cdot \frac{4034 + 2017}{2}$. The simplified quotient of these two

areas is the probability Laurent's number is larger than Chlo  's, which is $C = \frac{3}{4}$.

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2017 AMC 12A Problems/Problem 11

Problem

Claire adds the degree measures of the interior angles of a convex polygon and arrives at a sum of **2017**. She then discovers that she forgot to include one angle. What is the degree measure of the forgotten angle?

- (A)** 37 **(B)** 63 **(C)** 117 **(D)** 143 **(E)** 163

Solution

We know that the sum of the interior angles of the polygon is a multiple of **180**. Note that $\left\lceil \frac{2017}{180} \right\rceil = 12$ and $180 \cdot 12 = 2160$, so the angle Claire forgot is $\equiv 2160 - 2017 = 143 \pmod{180}$. Since the polygon is convex, the angle is ≤ 180 , so the answer is **(D) = 143**.

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2017 AMC 12A Problems/Problem 12

Problem

There are **10** horses, named Horse 1, Horse 2, ..., Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse ***k*** runs one lap in exactly ***k*** minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds. The least time ***S* > 0**, in minutes, at which all **10** horses will gain simultaneously be at the starting point is ***S* = 2520**. Let ***T* > 0** be the least time, in minutes, such that at least **5** of the horses are again at the starting point. What is the sum of the digits of ***T***?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution

We know that Horse ***k*** will be at the starting point after ***n*** minutes if ***k* | *n***. Thus, we are looking for the smallest ***n*** such that at least **5** of the numbers **{1, 2, ..., 10}** divide ***n***. Thus, ***n*** has at least **5** positive integer divisors.

We quickly see that **12** is the smallest number with at least **5** positive integer divisors, and that **1, 2, 3, 4, 6** are each numbers of horses. Thus, our answer is **1 + 2 = (B) 3**.

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2017 AMC 12A Problems/Problem 13

Problem

Driving at a constant speed, Sharon usually takes **180** minutes to drive from her house to her mother's house. One day Sharon begins the drive at her usual speed, but after driving $\frac{1}{3}$ of the way, she hits a bad snowstorm and reduces her speed by **20** miles per hour. This time the trip takes her a total of **276** minutes. How many miles is the drive from Sharon's house to her mother's house?

(A) 132 (B) 135 (C) 138 (D) 141 (E) 144

Solution

Let total distance be x . Her speed in miles per minute is $\frac{x}{180}$. Then, the distance that she drove before hitting the snowstorm is $\frac{x}{3}$. Her speed in snowstorm is reduced **20** miles per hour, or $\frac{1}{3}$ miles per minute. Knowing it took her **276** minutes in total, we create equation:

$$\text{Time before Storm} + \text{Time after Storm} = \text{Total Time} \Rightarrow$$

$$\frac{\text{Distance before Storm}}{\text{Speed before Storm}} + \frac{\text{Distance in Storm}}{\text{Speed in Storm}} = \text{Total Time} \Rightarrow \frac{\frac{x}{3}}{\frac{x}{180}} + \frac{\frac{2x}{3}}{\frac{x}{180} - \frac{1}{3}} = 276$$

Solving equation, we get $x = 135 \Rightarrow \boxed{B}$.

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2017 AMC 12A Problems/Problem 14

Problem

Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of **5** chairs under these conditions?

- (A) 12 (B) 16 (C) 28 (D) 32 (E) 40

Solution

Alice may sit in the center chair, in an end chair, or in a next-to-end chair. Suppose she sits in the center chair. The 2nd and 4th chairs (next to her) must be occupied by Derek and Eric, in either order, leaving the end chairs for Alice and Bob in either order; this yields $2! * 2! = 4$ ways to seat the group.

Next, suppose Alice sits in one of the end chairs. Then the chair beside her will be occupied by either Derek or Eric. The center chair must be occupied by Bob or Carla, leaving the last two people to fill the last two chairs in either order. 2 ways to seat Alice times 2 ways to fill the next chair times 2 ways to fill the center chair times 2 ways to fill the last two chairs yields $2 * 2 * 2 * 2 = 16$ ways to fill the chairs.

Finally, suppose Alice sits in the second or fourth chair. Then the chairs next to her must be occupied by Derek and Eric in either order, and the other two chairs must be occupied by Bob and Carla in either order. This yields $2 * 2 * 2 = 8$ ways to fill the chairs.

In total, there are $4 + 8 + 16$ ways to fill the chairs, so the answer is **(C) 28**.

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2017 AMC 12A Problems/Problem 15

Problem

Let $f(x) = \sin x + 2\cos x + 3\tan x$, using radian measure for the variable x . In what interval does the smallest positive value of x for which $f(x) = 0$ lie?

- (A) $(0, 1)$ (B) $(1, 2)$ (C) $(2, 3)$ (D) $(3, 4)$ (E) $(4, 5)$

Solution

We must first get an idea of what $f(x)$ looks like:

Between 0 and 1, $f(x)$ starts at 2 and increases; clearly there is no zero here.

Between 1 and $\frac{\pi}{2}$, $f(x)$ starts at a positive number and increases to ∞ ; there is no zero here either.

Between $\frac{\pi}{2}$ and 3, $f(x)$ starts at $-\infty$ and increases to some negative number; there is no zero here either.

Between 3 and π , $f(x)$ starts at some negative number and increases to -2; there is no zero here either.

Between π and $\pi + \frac{\pi}{4} < 4$, $f(x)$ starts at -2 and increases to

$$-\frac{\sqrt{2}}{2} + 2\left(-\frac{\sqrt{2}}{2}\right) + 3(1) = 3\left(1 - \frac{\sqrt{2}}{2}\right) > 0. \text{ There is a zero here by the Intermediate Value}$$

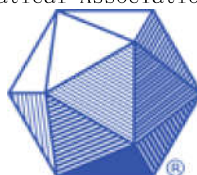
Theorem. Therefore, the answer is **(D)**.

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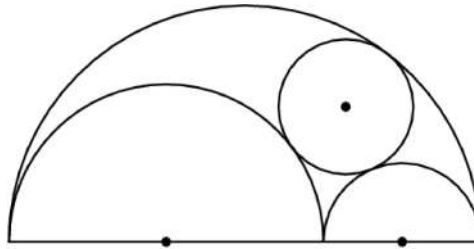
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Problem

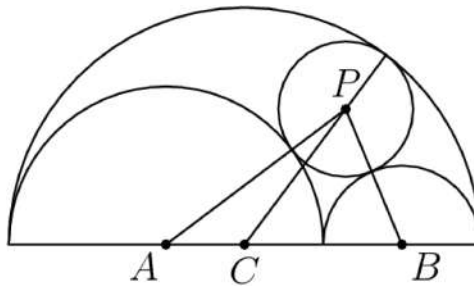
In the figure below, semicircles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semicircle with diameter JK . The two smaller semicircles are externally tangent to each other and internally tangent to the largest semicircle. A circle centered at P is drawn externally tangent to the two smaller semicircles and internally tangent to the largest semicircle. What is the radius of the circle centered at P ?



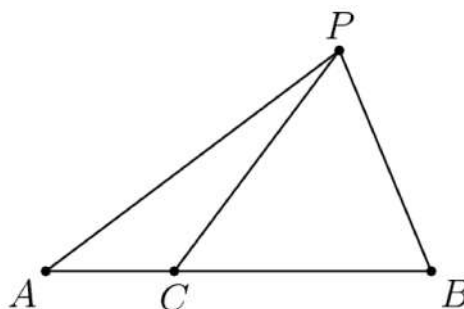
- (A) $\frac{3}{4}$ (B) $\frac{6}{7}$ (C) $\frac{1}{2}\sqrt{3}$ (D) $\frac{5}{8}\sqrt{2}$ (E) $\frac{11}{12}$

Solution 1

Connect the centers of the tangent circles! (call the center of the large circle C)



Notice that we don't even need the circles anymore; thus, draw triangle $\triangle ABP$ with cevian PC :



and use Stewart's Theorem:

$$AB \cdot AC \cdot BC + AB \cdot CP^2 = AC \cdot BP^2 + BC \cdot AP^2$$

From what we learned from the tangent circles, we have $AB = 3$, $AC = 1$, $BC = 2$, $AP = 2 + r$, $BP = 1 + r$, and $CP = 3 - r$, where r is the radius of the circle centered at P that we seek.

Thus:

$$3 \cdot 1 \cdot 2 + 3(3 - r)^2 = 1(1 + r)^2 + 2(2 + r)^2$$

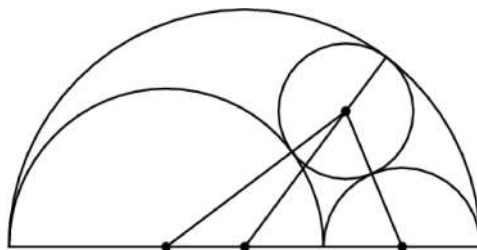
$$6 + 3(9 - 6r + r^2) = (1 + 2r + r^2) + 2(4 + 4r + r^2)$$

$$33 - 18r + 3r^2 = 9 + 10r + 3r^2$$

$$28r = 24$$

$$r = \boxed{\frac{6}{7}} \rightarrow \boxed{(B)}$$

Solution 2



Like the solution above, connecting the centers of the circles results in triangle $\triangle ABP$ with cevian PC . The two triangles $\triangle APC$ and $\triangle ABP$ share angle A , which means we can use Law of Cosines to set up a system of 2 equations that solve for r respectively:

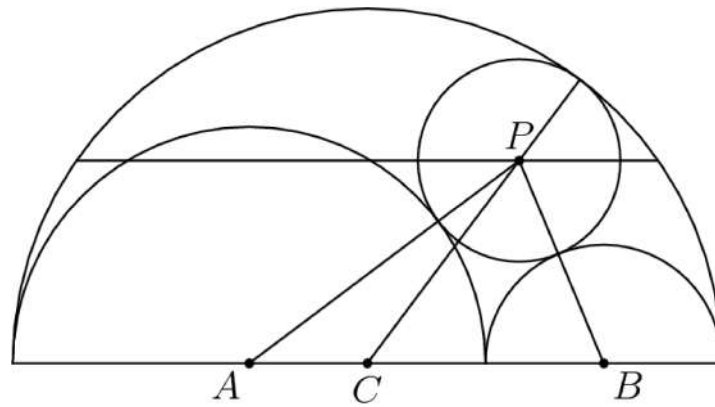
$$(2 + r)^2 + 1^2 - 2(2 + r)(1)\cos A = (3 - r)^2 \quad (\text{notice that the diameter of the largest semicircle is 6, so its radius is 3 and } PC \text{ is } 3 - r)$$

$$(2 + r)^2 + 3^2 - 2(2 + r)(3)\cos A = (r + 1)^2$$

We can eliminate the extra variable of angle A by multiplying the first equation by 3 and subtracting the second from it. Then, expand to find r :

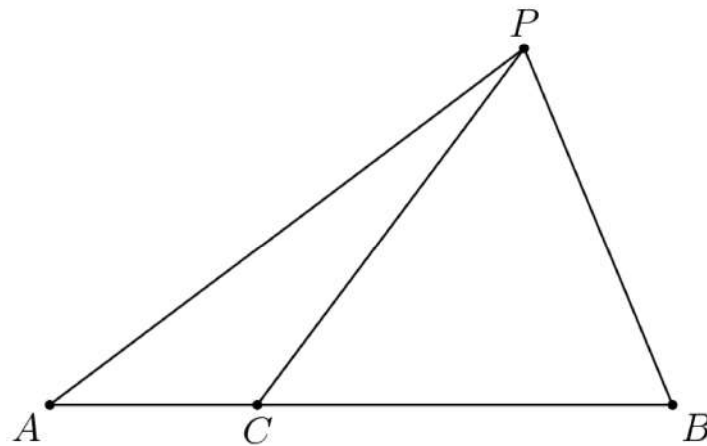
$$2(r^2 + 4r + 4) - 6 = 2r^2 - 20r + 26 \quad 8r + 2 = -20r + 26 \quad 28r = 24, \text{ so } r = 6/7 \quad \boxed{(B)}$$

Solution 3



Let C be the center of the largest semicircle and r be the radius of $\odot P$. We know that $AC = 1$, $CB = 2$, $AP = r + 2$, $BP = r + 1$, and $CP = 3 - r$. Notice that $\triangle ACP$ and $\triangle CBP$ are bounded by the same two parallel lines, so these triangles have the same heights. Because the bases of these two triangles (that have the same heights) differ by a factor of 2, the area of $\triangle CBP$ must be twice that of $\triangle ACP$, since the area of a triangle is $\frac{1}{2} \text{Base} \cdot \text{Height}$.

Again, we don't need to look at the circle and the semicircles anymore; just focus on the triangles.



Let A_1 equal to the area of $\triangle ACP$ and A_2 equal to the area of $\triangle CBP$. Heron's Formula states that the area of an triangle with sides a , b and c is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where s , or the semiperimeter, is $\frac{a+b+c}{2}$

The semiperimeter s_1 of $\triangle ACP$ is

$$\frac{[(r+2) + (3-r) + 1]}{2} = \frac{6}{2} = 3$$

Use Heron's Formula to obtain

$$A_1 = \sqrt{3(2)(3-2-r)(3-3+r)} = \sqrt{6r(1-r)} = \sqrt{6r-6r^2}$$

Using Heron's Formula again, find the area of $\triangle CBP$ with sides $r+1$, 2, and $3-r$.

$$s_2 = \frac{(r+1) + 2 + (3-r)}{2} = 3$$

$$A_2 = \sqrt{3(3-2)(3-1-r)(3-3+r)} = \sqrt{3(2r-r^2)} = \sqrt{6r-3r^2}$$

Now,

$$2 \cdot A_1 = A_2$$

$$2\sqrt{6r-6r^2} = \sqrt{6r-3r^2}$$

$$4(6r-6r^2) = 6r-3r^2$$

$$24r-24r^2 = 6r-3r^2$$

$$18r = 21r^2$$

$$r = \frac{18}{21} = \boxed{\frac{6}{7}} \rightarrow \boxed{\text{(B)}}$$

See Also

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2017 AMC 12A Problems/Problem 17

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Problem

There are **24** different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?

(A) 0 (B) 4 (C) 6 (D) 12 (E) 24

Solution 1

Note that these z such that $z^{24} = 1$ are $e^{\frac{ni\pi}{12}}$ for integer $0 \leq n < 24$. So

$$z^6 = e^{\frac{ni\pi}{2}}$$

This is real iff $\frac{n}{2} \in \mathbb{Z} \Leftrightarrow (n \text{ is even})$. Thus, the answer is the number of even $0 \leq n < 24$ which is

$$\boxed{(D) = 12}.$$

Solution 2

$$z = \sqrt[24]{1} = 1^{\frac{1}{24}}$$

By Euler's identity, $1 = e^{0 \times i} = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi)$, where k is an integer.

Using De Moivre's Theorem, we have $z = 1^{\frac{1}{24}} = \cos\left(\frac{k\pi}{12}\right) + i\sin\left(\frac{k\pi}{12}\right)$, where $0 \leq k < 24$ that produce 24 unique results.

$$\text{Using De Moivre's Theorem again, we have } z^6 = \cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)$$

For z^6 to be real, $\sin\left(\frac{k\pi}{2}\right)$ has to equal 0 to negate the imaginary component. This occurs whenever $\frac{k\pi}{2}$ is an integer multiple of π , requiring that k is even. There are exactly $\boxed{12}$ even values of k on the interval $0 \leq k < 24$, so the answer is $\boxed{(D)}$.

See Also

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2017 AMC 12A Problems/Problem 18

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Problem

Let $S(n)$ equal the sum of the digits of positive integer n . For example, $S(1507) = 13$. For a particular positive integer n , $S(n) = 1274$. Which of the following could be the value of $S(n+1)$?

(A) 1 (B) 3 (C) 12 (D) 1239 (E) 1265

Solution 1

Note that $n \equiv S(n) \pmod 9$, so $S(n+1) - S(n) \equiv n+1 - n = 1 \pmod 9$. So, since $S(n) = 1274 \equiv 5 \pmod 9$, we have that $S(n+1) \equiv 6 \pmod 9$. The only one of the answer choices $\equiv 6 \pmod 9$ is $\boxed{(D) = 1239}$.

Solution 2

One possible value of $S(n)$ would be 1275 , but this is not any of the choices. Ergo, we know that n ends in 9 , and after adding 1 , the last digit 9 carries over, turning the last digit into 0 . If the next digit is also a 9 , this process repeats until we get to a non- 9 digit. By the end, the sum of digits would decrease by 9 multiplied by the number of carry-overs but increase by 1 as a result of the final carrying over. Therefore, the result must be $9x - 1$ less than original value of $S(n)$, 1274 , where x is a positive integer. The only choice that satisfies this condition is $\boxed{1239}$, since $(1274 - 1239 + 1) \pmod 9 = 0$. The answer is $\boxed{D}$.

See Also

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2017 AMC 12A Problems/Problem 19

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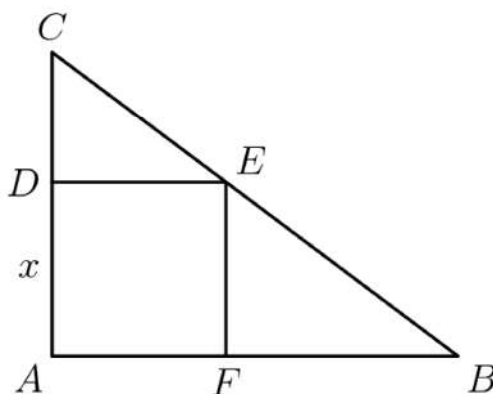
Problem

A square with side length x is inscribed in a right triangle with sides of length **3**, **4**, and **5** so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length **3**, **4**, and **5** so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

- (A) $\frac{12}{13}$ (B) $\frac{35}{37}$ (C) 1 (D) $\frac{37}{35}$ (E) $\frac{13}{12}$

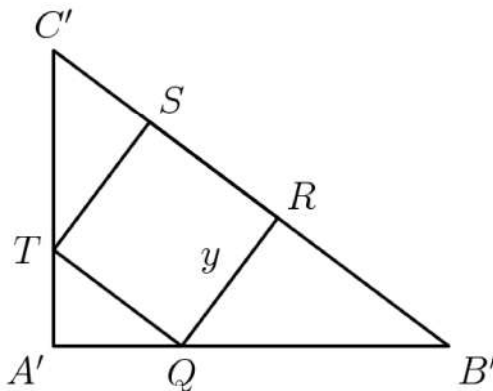
Solution 1

Analyze the first right triangle.



Note that $\triangle ABC$ and $\triangle FBE$ are similar, so $\frac{BF}{FE} = \frac{AB}{AC}$. This can be written as $\frac{4-x}{x} = \frac{4}{3}$. Solving, $x = \frac{12}{7}$.

Now we analyze the second triangle.



Similarly, $\triangle A'B'C'$ and $\triangle RB'Q$ are similar, so $RB' = \frac{4}{3}y$, and $C'S = \frac{3}{4}y$. Thus,
 $C'B' = C'S + SR + RB' = \frac{4}{3}y + y + \frac{3}{4}y = 5$. Solving for y , we get $y = \frac{60}{37}$. Thus, $\frac{x}{y} = \frac{37}{35}$.

Solution 2 (Shortcut using answer choices)

As stated in Solution 1, the square in the first triangle $\triangle ABC$ has a side length of $12/7$. Then we look at the answers. The only answer with a factor of 7 in the denominator is D .

See Also

2017 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2017))	
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2017 AMC 12A Problems/Problem 20

Problem

How many ordered pairs (a, b) such that a is a positive real number and b is an integer between **2** and **200**, inclusive, satisfy the equation $(\log_b a)^{2017} = \log_b(a^{2017})$?

(A) 198 (B) 199 (C) 398 (D) 399 (E) 597

Solution

By the properties of logarithms, we can rearrange the equation to read $2017 \log_b a = (\log_b a)^{2017}$. Then, subtracting $2017 \log_b a$ from each side yields $(\log_b a)^{2017} - 2017 \log_b a = 0$. We then proceed to factor out the term $\log_b a$ which results in $(\log_b a)((\log_b a)^{2016} - 2017) = 0$. Then, we set both factors equal to zero and solve.

$\log_b a = 0$ has exactly **199** solutions with the restricted domain of $[2, 200]$ since this equation will always have a solution in the form of $(1, b)$, and there are **199** possible values of b since $200 - 2 + 1 = 199$.

We proceed to solve the other factor, $(\log_b a)^{2016} - 2017$. We add **2017** to both sides, and take the **2016th** root, this gives us $\log_b a = \sqrt[2016]{2017}$ is a real number, and therefore $a = b^{\sqrt[2016]{2017}}$. Again, there are $199 \cdot 2 = 398$ solutions, as $b^{\sqrt[2016]{2017}}$ must be a real number (It's a real number raised to a real number), and we also have to multiply by two because when we take the **2016th** root, we must also consider the negative root which is valid because the taking the reciprocal of a negates $\log_b a$.

Therefore, the answer is $199 + 398 = \boxed{\text{(E) } 597}$.

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2017 AMC 12A Problems/Problem 21

Problem

A set S is constructed as follows. To begin, $S = \{0, 10\}$. Repeatedly, as long as possible, if x is an integer root of some polynomial $a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ for some $n \geq 1$, all of whose coefficients a_i are elements of S , then x is put into S . When no more elements can be added to S , how many elements does S have?

- (A) 4 (B) 5 (C) 7 (D) 9 (E) 11

Solution 1

At first, $S = \{0, 10\}$.

$10x + 10$	has root	$x = -1$	so now	$S = \{-1, 0, 10\}$
$-x^{10} - x^9 - x^8 - x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x + 10$	has root	$x = 1$	so now	$S = \{-1, 0, 1, 10\}$
$x + 10$	has root	$x = -10$	so now	$S = \{-10, -1, 0, 1, 10\}$
$x^4 - x^2 - x - 10$	has root	$x = 2$	so now	$S = \{-10, -1, 0, 1, 2, 10\}$
$x^4 - x^2 + x - 10$	has root	$x = -2$	so now	$S = \{-10, -2, -1, 0, 1, 2, 10\}$
$2x - 10$	has root	$x = 5$	so now	$S = \{-10, -2, -1, 0, 1, 2, 5, 10\}$
$2x + 10$	has root	$x = -5$	so now	$S = \{-10, -5, -2, -1, 0, 1, 2, 5, 10\}$

At this point, no more elements can be added to S . To see this, let

$$\begin{aligned} a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0 &= 0 \\ x(a_nx^{n-1} + a_{n-1}x^{n-2} + \dots + a_2x + a_1) + a_0 &= 0 \\ x(a_nx^{n-1} + a_{n-1}x^{n-2} + \dots + a_2x + a_1) &= -a_0 \end{aligned}$$

with each a_i in S . x is a factor of a_0 , and a_0 is in S , so x has to be a factor of some element in S . There are no such integers left, so there can be no more additional elements. $\{-10, -5, -2, -1, 0, 1, 2, 5, 10\}$ has 9 elements $\rightarrow$ **(D)**

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2017 AMC 12A Problems/Problem 22

Problem

A square is drawn in the Cartesian coordinate plane with vertices at $(2, 2)$, $(-2, 2)$, $(-2, -2)$, $(2, -2)$. A particle starts at $(0, 0)$. Every second it moves with equal probability to one of the eight lattice points (points with integer coordinates) closest to its current position, independently of its previous moves. In other words, the probability is $\frac{1}{8}$ that the particle will move from (x, y) to each of $(x, y + 1)$, $(x + 1, y + 1)$, $(x + 1, y)$, $(x + 1, y - 1)$, $(x, y - 1)$, $(x - 1, y - 1)$, $(x - 1, y)$, or $(x - 1, y + 1)$. The particle will eventually hit the square for the first time, either at one of the 4 corners of the square or at one of the 12 lattice points in the interior of one of the sides of the square. The probability that it will hit at a corner rather than at an interior point of a side is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Solution

We let c , e , and m be the probability of reaching a corner before an edge when starting at an "inside corner" (e.g. $(1, 1)$), an "inside edge" (e.g. $(1, 0)$), and the middle respectively.

Starting in the middle, there is a $\frac{4}{8}$ chance of moving to an inside edge and a $\frac{4}{8}$ chance of moving to an inside corner, so

$$m = \frac{1}{2}e + \frac{1}{2}c.$$

Starting at an inside edge, there is a $\frac{2}{8}$ chance of moving to another inside edge, a $\frac{2}{8}$ chance of moving to an inside corner, a $\frac{1}{8}$ chance of moving into the middle, and a $\frac{3}{8}$ chance of reaching an outside edge and stopping. Therefore,

$$e = \frac{1}{4}e + \frac{1}{4}c + \frac{1}{8}m + \frac{3}{8} * 0 = \frac{1}{4}e + \frac{1}{4}c + \frac{1}{8}m.$$

Starting at an inside corner, there is a $\frac{2}{8}$ chance of moving to an inside edge, a $\frac{1}{8}$ chance of moving into the middle, a $\frac{4}{8}$ chance of moving to an outside edge and stopping, and finally a $\frac{1}{8}$ chance of reaching that elusive outside corner. This gives

$$c = \frac{1}{4}e + \frac{1}{8}m + \frac{1}{2} * 0 + \frac{1}{8} * 1 = \frac{1}{4}e + \frac{1}{8}m + \frac{1}{8}.$$

Solving this system of equations gives

$$m = \frac{4}{35},$$

$$e = \frac{1}{14},$$

$$c = \frac{11}{70}.$$

Since the particle starts at $(0,0)$, it is m we are looking for, so the final answer is

$4 + 35 = \boxed{\text{(E)} 39}.$

See Also

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2017 AMC 12A Problems/Problem 23

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Problem

For certain real numbers a , b , and c , the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

(A) -9009 (B) -8008 (C) -7007 (D) -6006 (E) -5005

Solution

Let r_1, r_2 , and r_3 be the roots of $g(x)$. Let r_4 be the additional root of $f(x)$. Then from Vieta's formulas on the quadratic term of $g(x)$ and the cubic term of $f(x)$, we obtain the following:

$$\begin{aligned} r_1 + r_2 + r_3 &= -a \\ r_1 + r_2 + r_3 + r_4 &= -1 \end{aligned}$$

Thus $r_4 = a - 1$.

Now applying Vieta's formulas on the constant term of $g(x)$, the linear term of $g(x)$, and the linear term of $f(x)$, we obtain:

$$\begin{aligned} r_1 r_2 r_3 &= -10 \\ r_1 r_2 + r_2 r_3 + r_3 r_1 &= 1 \\ r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2 &= -100 \end{aligned}$$

Substituting for $r_1 r_2 r_3$ in the bottom equation and factoring the remainder of the expression, we obtain:

$$-10 + (r_1 r_2 + r_2 r_3 + r_3 r_1) r_4 = -10 + r_4 = -100$$

It follows that $r_4 = -90$. But $r_4 = a - 1$ so $a = -89$

Now we can factor $f(x)$ in terms of $g(x)$ as

$$f(x) = (x - r_4)g(x) = (x + 90)g(x)$$

Then $f(1) = 91g(1)$ and

$$g(1) = 1^3 - 89 \cdot 1^2 + 1 + 10 = -77$$

Hence $f(1) = 91 \cdot (-77) = \boxed{(C) - 7007}$.

Solution 2

Since all of the roots of $g(x)$ are distinct and are roots of $f(x)$, and the degree of f is one more than the degree of g , we have that

$$f(x) = C(x - k)g(x)$$

for some number k . By comparing x^4 coefficients, we see that $C = 1$. Thus,

$$x^4 + x^3 + bx^2 + 100x + c = (x - k)(x^3 + ax^2 + x + 10)$$

Expanding and equating coefficients we get that

$$a - k = 1, 1 - ak = b, 10 - k = 100, -10k = c$$

The third equation yields $k = -90$, and the first equation yields $a = -89$. So we have that

$$f(1) = (1 + 90)g(1) = 91(1 - 89 + 1 + 10) = (91)(-77) = \boxed{(C) - 7007}$$

See Also

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2017 AMC 12A Problems/Problem 24

Problem

Quadrilateral $ABCD$ is inscribed in circle O and has side lengths $AB = 3$, $BC = 2$, $CD = 6$, and $DA = 8$. Let X and Y be points on

$\overline{BD}$ such that $\frac{DX}{BD} = \frac{1}{4}$ and $\frac{BY}{BD} = \frac{11}{36}$. Let E be the intersection of line AX and the line through Y parallel to $\overline{AD}$. Let F be the intersection of line CX and the line through E parallel to $\overline{AC}$. Let G be the point on circle O other than C that lies on line CX . What is $XF \cdot XG$?

- (A) 17 (B) $\frac{59 - 5\sqrt{2}}{3}$ (C) $\frac{91 - 12\sqrt{3}}{4}$ (D) $\frac{67 - 10\sqrt{2}}{3}$ (E) 18

Solution

It is easy to see that $\frac{XY}{BD} = 1 - \frac{1}{4} - \frac{11}{36} = \frac{4}{9}$. First we note that $\triangle AXD \sim \triangle EXY$ with a ratio of $\frac{DX}{XY} = \frac{9}{16}$. Then $\triangle ACX \sim \triangle EFX$ with a ratio of $\frac{AX}{XE} = \frac{DX}{XY} = \frac{9}{16}$, so $\frac{XF}{CX} = \frac{16}{9}$. Now we find the length of BD . Because the quadrilateral is cyclic, we can simply use the Law of Cosines.

$$BD^2 = 3^2 + 8^2 - 48 \cos \angle BAD = 2^2 + 6^2 - 24 \cos(180 - \angle BAD) = 2^2 + 6^2 + 24 \cos \angle BAD$$

$$\rightarrow \cos \angle BAD = \frac{11}{24}$$

$$\rightarrow BD = \sqrt{51}$$

By Power of a Point, $CX \cdot XG = DX \cdot XB = \frac{\sqrt{51}}{4} \cdot \frac{3\sqrt{51}}{4}$. Thus

$$XF \cdot XG = \frac{XF}{CX} CX \cdot XG = \frac{51}{3} = \boxed{\text{(A) } 17}.$$

—solution by FRaelya

See Also

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2017 AMC 12A Problems/Problem 25

Problem

The vertices V of a centrally symmetric hexagon in the complex plane are given by

$$V = \left\{ \sqrt{2}i, -\sqrt{2}i, \frac{1}{\sqrt{8}}(1+i), \frac{1}{\sqrt{8}}(-1+i), \frac{1}{\sqrt{8}}(1-i), \frac{1}{\sqrt{8}}(-1-i) \right\}.$$

For each j , $1 \leq j \leq 12$, an element z_j is chosen from V at random, independently of the other choices. Let $P = \prod_{j=1}^{12} z_j$ be the product of the 12 numbers selected. What is the probability that $P = -1$?

(A) $\frac{5 \cdot 11}{3^{10}}$ (B) $\frac{5^2 \cdot 11}{2 \cdot 3^{10}}$ (C) $\frac{5 \cdot 11}{3^9}$ (D) $\frac{5 \cdot 7 \cdot 11}{2 \cdot 3^{10}}$ (E) $\frac{2^2 \cdot 5 \cdot 11}{3^{10}}$

Solution

It is possible to solve this problem using elementary counting methods. This solution proceeds by a cleaner generating function.

We note that $\pm\sqrt{2}i$ both lie on the imaginary axis and each of the $\frac{1}{\sqrt{8}}(\pm 1 \pm i)$ have length $\frac{1}{2}$ and angle of odd multiples of $\pi/4$, i.e. $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. When we draw these 6 complex numbers out on the complex plane, we get a crystal-looking thing. Note that the total number of ways to choose 12 complex numbers is 6^{12} . Now we count the number of good combinations.

We first consider the lengths. When we multiply 12 complex numbers together, their magnitudes multiply. Suppose we have n of the numbers $\pm\sqrt{2}i$; then we must have $(\sqrt{2})^n \cdot \left(\frac{1}{2}\right)^{12-n} = 1 \implies n = 8$. Having $n = 8$ will take care of the length of the product; now we need to deal with the angle.

We require $\sum \theta \equiv \pi \pmod{2\pi}$. Letting z be $e^{i\pi/4}$, we see that the angles we have available are $\{z^1, z^2, z^3, z^5, z^6, z^7\}$, where we must choose exactly 8 angles from the set $\{z^2, z^6\}$ and exactly 4 from the set $\{z^1, z^3, z^5, z^7\}$. If we found a good combination where we had a_i of each angle z^i , then the amount this would contribute to our count would be $\binom{12}{4, 8} \cdot \binom{8}{a_2, a_6} \cdot \binom{4}{a_1, a_3, a_5, a_7}$. We want to add these all up. We proceed by generating functions.

Consider

$$(t_2x^2 + t_6x^6)^8(t_1x^1 + t_3x^3 + t_5x^5 + t_7x^7)^4.$$

The expansion will be of the form $\sum_i \left(\sum_{\sum a=i} \binom{8}{a_2, a_6} \binom{4}{a_1, a_3, a_5, a_7} t_1^{a_1} t_2^{a_2} t_3^{a_3} t_5^{a_5} t_6^{a_6} t_7^{a_7} x^i \right)$.

Note that if we reduced the powers of $x \pmod 8$ and fished out the coefficient of x^4 and plugged in $t_i = 1 \forall i$ (and then multiplied by $\binom{12}{4, 8}$) then we would be done. Since plugging in $t_i = 1$ doesn't affect the x 's, we do that right away. The expression then becomes

$$x^{20}(1+x^4)^8(1+x^2+x^4+x^6)^4 = x^{20}(1+x^4)^{12}(1+x^2)^4 = x^4(1+x^4)^{12}(1+x^2)^4,$$

where the last equality is true because we are taking the powers of $x \pmod 8$. Let $[x^n]f(x)$ denote the coefficient of x^n in $f(x)$. Note $[x^4]x^4(1+x^4)^{12}(1+x^2)^4 = [x^0](1+x^4)^{12}(1+x^2)^4$. We use the roots of unity filter, which states

terms of $f(x)$ that have exponent congruent to $k \bmod n = \frac{1}{n} \sum_{m=1}^n \frac{f(z^m x)}{z^{mk}}$,

where $z = e^{i\pi/n}$. In our case $k = 0$, so we only need to find the average of the $f(z^m x)$ s.

$$\begin{aligned} z^0 &\Rightarrow (1+x^4)^{12}(1+x^2)^4, \\ z^1 &\Rightarrow (1-x^4)^{12}(1+ix^2)^4, \\ z^2 &\Rightarrow (1+x^4)^{12}(1-x^2)^4, \\ z^3 &\Rightarrow (1-x^4)^{12}(1-ix^2)^4, \\ z^4 &\Rightarrow (1+x^4)^{12}(1+x^2)^4, \\ z^5 &\Rightarrow (1-x^4)^{12}(1+ix^2)^4, \\ z^6 &\Rightarrow (1+x^4)^{12}(1-x^2)^4, \\ z^7 &\Rightarrow (1-x^4)^{12}(1-ix^2)^4. \end{aligned}$$

We plug in $x = 1$ and take the average to find the sum of all coefficients of $x^{\text{multiple of } 8}$. Plugging in $x = 1$ makes all of the above zero except for z^0 and z^4 . Averaging, we get 2^{14} . Now the answer is simply

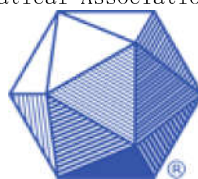
$$\frac{\binom{12}{4,8}}{6^{12}} \cdot 2^{14} = \boxed{\frac{2^2 \cdot 5 \cdot 11}{3^{10}}}.$$

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