2023 AIME II 卷真题

Problem 1:

The numbers of apples growing on each of six apple trees form an arithmetic sequence where the greatest number of apples growing on any of the six trees is double the least number of apples growing on any of the six trees. The total number of apples growing on all six trees is 990. Find the greatest number of apples growing on any of the six trees.

有六棵苹果树，上面所结苹果的数量构成了一个等差数列，其中最大数量是最小数量的两倍，六棵树总共结了 990 颗苹果。请问这六棵树中苹果最多的结了多少颗苹果？

Problem 2:

Recall that a palindrome is a number that reads the same forward and backward. Find the greatest integer less than 1000 that is a palindrome both when written in base ten and when written in base eight, such as $292 = 444_{\text{eight}}$.

“回文数”指的是反向排列与原数相同的数字。请找到比 1000 小的最大的数字，使得其在十进制和八进制下都是回文数。如$292 = 444_8$，

Problem 3:

Let $\triangle ABC$ be an isosceles triangle with $\angle A = 90°$. There exists a point P inside ΔABC such that $\angle PAB = \angle PBC = \angle PCA$ and $AP = 10$. Find the area of ΔABC.

$\triangle ABC$是等腰三角形且$\angle A = 90°$。在三角形内部存在一点P，使得$\angle PAB = \angle PBC = \angle PCA$且$AP = 10$。求$\triangle ABC$的面积。

Problem 4:

Let x, y, and z be real numbers satisfying the system of equations

$$xy + 4z = 60$$
$$yz + 4x = 60$$
$$zx + 4y = 60.$$

Let S be the set of possible values of x. Find the sum of the squares of the elements of S.

x、y、z均为实数且满足下列方程组：

$$xy + 4z = 60$$
$$yz + 4x = 60$$
$$zx + 4y = 60$$

令S为x所有可能值的集合，求S中所有元素的平方和。

Problem 5:

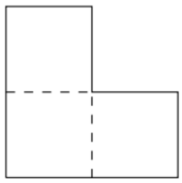
Let S be the set of all positive rational numbers r such that when the two numbers r and $55r$ are written as fractions in lowest terms, the sum of the numerator and denominator of one fraction is the same as the sum of the numerator and denominator of the other fraction. The sum of all the elements of S can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

S是一个正有理数集合，其中只包含这样的有理数r：当r和$55r$都表示成最简形式的分数时，其中一个分数的分子与分母的和等于另一个分数分子与分母的和。S中所有元素的和可以表示成$\frac{p}{q}$，其中p和q是互质的正整数。求$p + q$的值。

Problem 6:

Consider the L-shaped region formed by three unit squares joined al their sides, as shown below. Two points A and B are chosen independently and uniformly at random from inside this region. The probability that the midpoint of $\overline{AB}$ also lies inside this L-shaped region can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

如图，三个单位正方形邻边重合构成“L”区域。点A和点B独立且随机地从这个区域内部进行选取。线段AB的中点也在“L”区域内部的概率为$\frac{m}{n}$，其中m和n是互质的正整数。求$m+n$的值。



Problem 7:

Each vertex of a regular dodecagon (12-gon) is to be colored either red or blue, and thus there are 2^{12} possible colorings. Find the number of these colorings with the property that no four vertices colored the same color are the four vertices of a rectangle.

一个正 12 边形的所有顶点都要被涂成红色或者蓝色，这样的话就总共有2^{12}种涂色方案。如果这些顶点中的任意四个点构成了长方形，那么这四个顶点的颜色不能全部相同，请问这样的涂色方案总共有多少种？

Problem 8:

Let $\omega = cos\frac{2\pi}{7} + i \cdot \sin\frac{2\pi}{7}$, where $i = \sqrt{-1}$. Find the value of the product

$$\prod_{k=0}^{6}(\omega^{3k} + \omega^{k} + 1)$$

令$\omega = \cos\frac{2\pi}{7} + i \cdot \sin\frac{2\pi}{7}$，其中$i = \sqrt{-1}$。求下列乘积的值：

$$\prod_{k=0}^{6}(\omega^{3k} + \omega^{k} + 1)$$

Problem 9:

Circles ω_1 and ω_2 intersect at two points P and Q, and their common tangent line closer to P intersects ω_1 and ω_2 at points A and B, respectively. The line parallel to line AB that passes through P intersects ω_1 and ω_2 for the second time at points X and Y, respectively. Suppose $PX = 10$, $PY = 14$, and $PQ = 5$. Then the area of trapezoid $XABY$ is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

圆ω_1和ω_2相交于点P和点Q，且两圆在P点一侧的公切线与ω_1和ω_2分别交于A点和B点。经过P点且平行于直线AB的直线分别交ω_1和ω_2于点X和点Y。若$PX = 10$，$PY = 14$，$PQ = 5$，则梯形$XABY$的面积可以表示为$m\sqrt{n}$，其中m和n都是正整数，且n中不含有能开方的因数。求$m + n$的值。

Problem 10:

Let N be the number of ways to place the integers 1 through 12 in the 12 cells of a 2×6 grid so that for any two cells sharing a side, the difference between the numbers in those cells is not divisible by 3. One way to do this is shown below. Find the number of positive integer divisors of N.

设N为将整数 1~12 填入2×6方格中，且保证相邻方格中的数的差不能被 3 整除的填法种数。其中一种满足条件的填法如下图。求N的正因数的个数。

1	3	5	7	9	11
2	4	6	8	10	12

Problem 11:

Find the number of collections of 16 distinct subsets of $\{1, 2, 3, 4, 5\}$ with the property that for any two subsets X and Y in the collection, $X \cap Y \neq \emptyset$.

在{1,2,3,4,5}的全部子集中挑选 16 个形成新的集合，求满足如下条件的这样的集合个数：对任何该集合中的集合X和Y，$X \cap Y \neq \emptyset$

Problem 12:

In $\triangle ABC$ with side lengths $AB = 13$, $BC = 14$, and $CA = 15$, let M be the midpoint of $\overline{BC}$. Let P be the point on the circumcircle of $\triangle ABC$ such that M is on $\overline{AP}$. There exists a unique point Q on segment $\overline{AM}$ such that $\angle PBQ = \angle PCQ$. Then AQ can be written as $\frac{m}{\sqrt{n}}$, where m and n are relatively prime positive integers. Find $m + n$.

在$\triangle ABC$中，$AB = 13$，$BC = 14$，$CA = 15$，M为线段BC的中点。令P点为$\triangle ABC$外接圆上的一点，且是的M点在线段AP上。在线段AM上存在唯一点Q，使得$\angle PBQ = \angle PCQ$。那么AQ的长度可以被写作$\frac{m}{\sqrt{n}}$，其中m和n是互质的正整数。求$m + n$的值。

Problem 13:

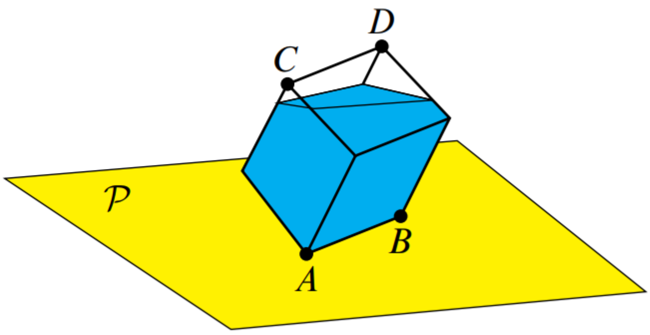
Let A be an acute angle such that $\tan A = 2\cos A$. Find the number of positive integers n less than or equal to 1000 such that $\sec^n A + \tan^n A$ is a positive integer whose units digit is 9.

A为一个锐角，且$\tan A = 2\cos A$。求小于等于 1000 且满足条件的n的个数：$\sec^n A + \tan^n A$是一个个位数为 9 的正整数。

Problem 14:

A cube-shaped container has vertices *A, B, C, and D*, where $\overline{AB}$ and $\overline{CD}$ are parallel edges of the cube, and $\overline{AC}$ and $\overline{BD}$ are diagonals of faces of the cube, as shown. Vertex A of the cube is set on a horizontal plane $\mathcal{P}$ so that the plane of the rectangle $ABDC$ is perpendicular to $\mathcal{P}$, vertex B is 2 meters above $\mathcal{P}$, vertex C is 8 meters above $\mathcal{P}$, and vertex D is 10 meters above $\mathcal{P}$. The cube contains water whose surface is parallel to $\mathcal{P}$ at a height of 7 meters above $\mathcal{P}$. The volume of the water is $\frac{m}{n}$ cubic meters, where m and n are relatively prime positive integers. Find $m+n$.

如图，A、B、C、D是一个立方体形状的容器的四个顶点，其中线段AB和CD是平行的，线段AC和线段BD是面对角线。顶点A是水平面$\mathcal{P}$,中的一点，且长方形$ABDC$是垂直于$\mathcal{P}$,的，点B在$\mathcal{P}$,上方 2 米，点C在$\mathcal{P}$,上方 8 米，点D在$\mathcal{P}$,上方 10 米。这个容器中的水平面是平行于$\mathcal{P}$,的，且比$\mathcal{P}$,高了 7 米。其中水的体积可以表示为$\frac{m}{n}$立方米，其中m和n是互质的正整数。求$m+n$的值。



Problem 15:

For each positive integer n let a_n be the least positive integer multiple of 23 such that $a_n \equiv 1 (\text{mod } 2^n)$. Find the number of positive integers n less than or equal to 1000 that satisfy $a_n = a_{n+1}$

对于每一个正整数 n，令a_n为 23 最小的倍数能使得$a_n \equiv 1 (\text{mod } 2^n)$。求小于等于 1000 且使得$a_n = a_{n+1}$的 n 的个数。