2021Spring AMC 12A

Problem 1

What is the value of $2^{1+2+3}-(2^1+2^2+2^3)$?

下式的值是多少

$$2^{1+2+3}-(2^1+2^2+2^3)?$$

(A) 0 **(B)** 50 **(C)** 52 **(D)** 54 **(E)** 57

Problem 2

Under what conditions does $\sqrt{a^2+b^2}=a+b$ hold, where a and b are real numbers?

已知 *a* 和 *b* 是实数，问在何种情况下，$\sqrt{a^2+b^2}=a+b$成立？

(A) It is never true.丨永远不成立。

(B) It is true if and only if *ab*=0.丨当且仅当 *ab*=0 时才成立。

(C) It is true if and *a*+*b*⩾0.丨当且仅当 *a*+*b*⩾0 时才成立。

(D) It is true if and only if *ab*=0 and *a*+*b*⩾0.| 当且仅当 *ab*=0 且 *a*＋*b*⩾0 时时才成立。

(E) It is always true.丨总是成立。

Problem 3

The sum of two natural numbers is $17{,}402$. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

两个自然数之和是 17,402。这两个数中的一个可以被 10 整除。如果去掉该数的个位数字则得到另外一个数。问这两个数的差是多少？

(A) 10,272 **(B)** 11,700 **(C)** 13,362 **(D)** 14,238 **(E)** 15,426

Problem 4

Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that

all of his happy snakes can add,

none of his purple snakes can subtract, and

all of his snakes that can't subtract also can't add.

Which of these conclusions can be drawn about Tom's snakes?

汤姆有 13 条蛇，其中 4 条是紫色的，5 条是快乐的。他观察发现

- 他的所有快乐的蛇都能做加法，
- 他的紫色的蛇不会做减法，而且
- 他所有不会做减法的蛇也不会做加法。

关于汤姆的蛇，可以得出以下哪个结论？

（A） Purple snakes can add. | 紫色的蛇可以做加法。

（B） Purple snakes are happy. | 紫色的蛇是快乐的。

（C） Snakes that can add are purple. | 能做加法的蛇是紫色的。

（D） Happy snakes are not purple. | 快乐的蛇不是紫色的。

（E） Happy snakes can’t subtract. | 快乐的蛇不会做减法。

Problem 5

When a student multiplied the number 66 by the repeating decimal, $\underline{1}.\underline{a}\ \underline{b}\ \underline{a}\ \underline{b}\ldots=\underline{1}.\overline{\underline{a}\ \underline{b}},$ where a and b are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{a}\ \underline{b}.$ Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit number $\underline{a}\ \underline{b}$?

一名学生在用 66 乘以如下的循环小数时，$\underline{1}.\underline{a}\ \underline{b}\ \underline{a}\ \underline{b}\ldots=\underline{1}.\overline{\underline{a}\ \underline{b}},$其中 a 和 b 是数字，他没有注意到循环小数标识，而只是做了 66 乘以$\underline{1}.\underline{a}\ \underline{b}.$。后来他发现他的答案比正确答案小 0.5。

问 2 位整数$\underline{a}\ \underline{b}$是多少?

$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 45 \qquad \textbf{(D)}\ 60 \qquad \textbf{(E)}\ 75$

Problem 6

A deck of cards has only red cards and black cards. The probability of a randomly chosen card being red is $\frac{1}{3}$. When 4 black cards are added to the deck, the probability of choosing red becomes $\frac{1}{4}$. How many cards were in the deck originally?

一副牌中只有红色卡片和黑色卡片。随机选出一张卡片是红色的概率为$\frac{1}{3}$，当这副牌中增加 4 张黑色卡片后，选出一张牌是红色的概率变为$\frac{1}{4}$。问原本这副牌中有多少张卡片？

$\textbf{(A) } 6 \qquad \textbf{(B) } 9 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 15 \qquad \textbf{(E) } 18$

Problem 7

What is the least possible value of $(xy-1)^2+(x+y)^2$ for all real numbers x and y?

对于实数 *x* 和 *y*，$(xy-1)^2+(x+y)^2$的最小可能值是多少？

$\textbf{(A) } 0 \qquad \textbf{(B) } \frac{1}{4} \qquad \textbf{(C) } \frac{1}{2} \qquad \textbf{(D) } 1 \qquad \textbf{(E) } 2$

Problem 8

A sequence of numbers is defined by $D_0=0, D_1=0, D_2=1$ and $D_n=D_{n-1}+D_{n-3}$ for $n\ge 3$. What are the parities (evenness or oddness) of the triple of numbers $(D_{2021},D_{2022},D_{2023})$, where E denotes even and O denotes odd?

一个数列按如下方式定义：$D_0=0, D_1=0, D_2=1$并且对于$n\ge 3$, $D_n=D_{n-1}+D_{n-3}$。

求三元数组$(D_{2021},D_{2022},D_{2023})$的奇偶性，这里用 *E* 表示偶数，用 *O* 表示奇数。

$\textbf{(A) } (O,E,O) \qquad \textbf{(B) } (E,E,O) \qquad \textbf{(C) } (E,O,E) \qquad \textbf{(D) } (O,O,E) \qquad \textbf{(E) } (O,O,O)$

Problem 9

Which of the following is equivalent to

$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})$?

下列算式与下面哪个表达式相等

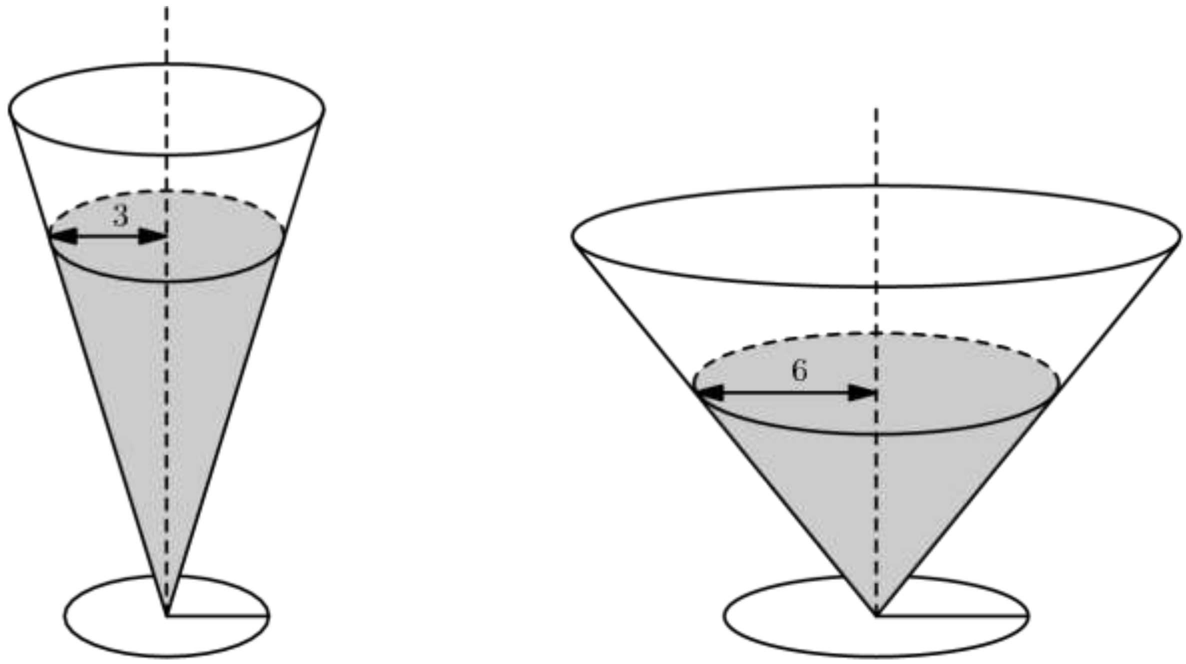
$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})$?

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$ **(D)** $3^{128}+2^{128}$ **(E)** 5^{127}

Problem 10

Two right circular cones with vertices facing down as shown in the figure below contains the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?

如图所示，两个顶点朝下的正圆锥包含相同量的液体。液体顶部表面的半径分别为 3 厘米和 6 厘米。在每个圆锥体中放入一个半径为 1 厘米的球形弹子，它沉入底部，完全浸没，没有任何液体溢出。问窄圆锥内液面上升的高度与宽圆锥内液面上升的高度之比是多少？



(A) $1:1$ **(B)** $47:43$ **(C)** $2:1$ **(D)** $40:13$ **(E)** $4:1$

Problem 11

A laser is placed at the point $(3,5)$. The laser beam travels in a straight line. Larry wants the beam to hit and bounce off the y-axis, then hit and bounce off the x-axis, then hit the point $(7,5)$. What is the total distance the beam will travel along this path?

在点（3，5）处放置有一个激光发射器。激光束沿直线传播。Larry 想让光束打在 *y*-轴上并反射，然后打在 *x*-轴上并反射，之后打到点（7，5）。问光束沿着这条路径所经过的总距离是多少？

$\textbf{(A)}\ 2\sqrt{10} \qquad \textbf{(B)}\ 5\sqrt2 \qquad \textbf{(C)}\ 10\sqrt2 \qquad \textbf{(D)}\ 15\sqrt2 \qquad \textbf{(E)}\ 10\sqrt5$

Problem 12

All the roots of the polynomial $z^6-10z^5+Az^4+Bz^3+Cz^2+Dz+16$ are positive integers, possibly repeated. What is the value of B?

多项式$z^6-10z^5+Az^4+Bz^3+Cz^2+Dz+16$的根都是正整数，有可能重复。问 *B* 的取值是多少？

$\textbf{(A)}\ -88 \qquad \textbf{(B)}\ -80 \qquad \textbf{(C)}\ -64 \qquad \textbf{(D)}\ -41 \qquad \textbf{(E)}\ -40$

Problem 13

Of the following complex numbers z, which one has the property that z^5 has the greatest real part?

在下面的各个复数 *z* 中，哪一个数使得z^5的实数部分最大？

$\textbf{(A)}\ -2 \qquad \textbf{(B)}\ -\sqrt3+i \qquad \textbf{(C)}\ -\sqrt2+\sqrt2 i \qquad \textbf{(D)}\ -1+\sqrt3 i \qquad \textbf{(E)}\ 2i$

Problem 14

What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

下列算式的值是多少

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

$\textbf{(A) }21 \qquad \textbf{(B) }100\log_5 3 \qquad \textbf{(C) }200\log_3 5 \qquad \textbf{(D) }2{,}200 \qquad \textbf{(E) }21{,}000$

Problem 15

A choir direction must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the numbers of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of different groups that could be selected. What is the remainder when N is divided by 100?

合唱团指挥必须从他的 6 名男高音和 8 名男低音中挑选一组歌手。仅有的要求是男高音与男低音人数之差必须是 4 的倍数，并且至少要有一名歌手。假设 *N* 是组成这样一组歌手的方法数。当 *N* 除以 100 时，余数是多少？

$\textbf{(A) } 47 \qquad \textbf{(B) } 48 \qquad \textbf{(C) } 83 \qquad \textbf{(D) } 95 \qquad \textbf{(E) } 96$

Problem 16

In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.
$1, 2, 2, 3, 3, 3, 4, 4, 4, ..., 200, 200, ..., 200$What is the median of the numbers in this list?

在下面的数据列表中，对于 1≤*n*≤200，整数 *n* 出现了 *n* 次。
$1, 2, 2, 3, 3, 3, 4, 4, 4, ..., 200, 200, ..., 200$

问这组数据列表的中位数是多少？

$\textbf{(A)} ~100.5 \qquad \textbf{(B)} ~134 \qquad \textbf{(C)} ~142 \qquad \textbf{(D)} ~150.5 \qquad \textbf{(E)} ~167$

Problem 17

Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}, BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length of AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

在梯形 *ABCD* 中，$\overline{AB} \parallel \overline{CD}, BC = CD = 43$，并且$\overline{AD} \perp \overline{BD}$。设 *O* 是对角线$\overline{AC}$和$\overline{BD}$的交点，*P* 是$\overline{BD}$的中点。已知 *OP*＝11，*AD* 的长度可以表示成$m\sqrt{n}$，其中 *m* 和 *n* 是正整数，并且 *n* 不能被任何质数的平方所整除。问 *m*＋*n* 的值是多少？

$\textbf{(A)}\ 65 \qquad \textbf{(B)}\ 132 \qquad \textbf{(C)}\ 157 \qquad \textbf{(D)}\ 194 \qquad \textbf{(E)}\ 215$

Problem 18

Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Furthermore, suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

令 *f* 是一个定义在正有理数集合上的函数，它具有性质：对于所有的正有理数 *a* 和 *b*，$f(a \cdot b) = f(a) + f(b)$。假设 *f* 还具有性质：对于每一个质数$f(p) = p$。问以下哪个数 *x*，满足$f(x) < 0$？

$\textbf{(A)}\ \frac{17}{32} \qquad \textbf{(B)}\ \frac{11}{16} \qquad \textbf{(C)}\ \frac{7}{9} \qquad \textbf{(D)}\ \frac{7}{6} \qquad \textbf{(E)}\ \frac{25}{11}$

Problem 19

How many solutions does the equation $\sin\left(\frac{\pi}{2}\cos x\right)=\cos\left(\frac{\pi}{2}\sin x\right)$ have in the closed interval $[0,\pi]$?

方程$\sin\left(\frac{\pi}{2}\cos x\right)=\cos\left(\frac{\pi}{2}\sin x\right)$在闭区间$[0,\pi]$有多少个解？

$\textbf{(A) }0 \qquad \textbf{(B) }1 \qquad \textbf{(C) }2 \qquad \textbf{(D) }3 \qquad \textbf{(E) }4$

Problem 20

Suppose that on the parabola with vertex V and a focus F there exists a point A such that $AF=20$ and $AV=21$. What is the sum of all possible values of the length FV?

假设在顶点为 *V* 和焦点为 *F* 的抛物线上存在一个点 *A*,使得 *AF*＝20，*AV*=21，*FV* 的长度的所有可能值的总和是多少？

$\textbf{(A) }13 \qquad \textbf{(B) }\frac{40}{3} \qquad \textbf{(C) }\frac{41}{3} \qquad \textbf{(D) }14 \qquad \textbf{(E) }\frac{43}{3}$

Problem 21

The five solutions to the equation $(z-1)(z^2+2z+4)(z^2+4z+6)=0$ may be written in the form x_k+y_ki for $1\le k\le 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points $(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$, and (x_5,y_5). The eccentricity of $\mathcal{E}$ can be written in the form $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. What is $m+n$? (Recall that the eccentricity of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of $\mathcal{E}$ and $2c$ is the is the distance between its two foci.)

方程$(z-1)(z^2+2z+4)(z^2+4z+6)=0$的五个解可以写成$x_k+y_ki$的形式，其中 1≤*k*≤5，$x_k$，$y_k$是实数。令$\mathcal{E}$是唯一的通过点$(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$,和$(x_5,y_5)$的椭圆。$\mathcal{E}$的偏心率可以写成$\sqrt{\frac{m}{n}}$的形式，其中 *m* 和 *n* 是互质的正整数。问$m+n$是多少？（注意，椭圆$\mathcal{E}$的偏心率是指比率$\frac{c}{a}$，其中 2*a* 是椭圆$\mathcal{E}$长轴的长度，而 2*c* 是两个焦点之间的距离。）

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

Problem 22

Suppose that the roots of the polynomial $P(x)=x^3+ax^2+bx+c$ are $\cos\frac{2\pi}{7},\cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where angles are in radians. What is abc?

假设多项式$P(x)=x^3+ax^2+bx+c$的根是$\cos\frac{2\pi}{7},\cos\frac{4\pi}{7},\cos\frac{6\pi}{7}$，其中角度按弧度制表示。问 *abc* 是多少？

(A) $-\frac{3}{49}$ **(B)** $-\frac{1}{28}$ **(C)** $\frac{\sqrt[3]{7}}{64}$ **(D)** $\frac{1}{32}$ **(E)** $\frac{1}{28}$

Problem 23

Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

青蛙 Frieda 在一个 3 x 3 的方格表上开始一系列跳跃，每次跳跃都随机选择一个方向——向上，向下，向左或向右，从一个方格移动到旁边的方格。她不能斜着跳。当跳跃的方向会使得 Frieda 离开方格表时，她会“绕个圈”，跳到相对的另一边。例如，如果 Frieda 从中心方格开始，向上跳跃两次，第一次跳跃后她将位于最上面一行的中间方格，第二次跳跃将使得 Frieda 跳到相对的边，落在最下面一行的中间方格。假设 Frieda 从中心方格出发，最多随机跳跃四次，并且当到达角落方格时就停止跳跃。问她在四次跳跃中到达角落方格的概率是多少？

$\textbf{(A)}\ \frac{9}{16} \qquad \textbf{(B)}\ \frac{5}{8} \qquad \textbf{(C)}\ \frac{3}{4} \qquad \textbf{(D)}\ \frac{25}{32} \qquad \textbf{(E)}\ \frac{13}{16}$

Problem 24

Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, then the area of $\triangle PQR$ equals $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. What is $a+b+c$?

半圆Γ直径$\overline{AB}$的长度为 14。圆Ω与$\overline{AB}$相切于点 *P* 并与Γ相交于点 *Q* 和 *R*。如果 $QR = 3\sqrt{3}$，并且$\angle QPR = 60^\circ$，那么 $\triangle PQR$的面积是 $\frac{a\sqrt{b}}{c}$，其中 *a* 和 *c* 是互质的正整数，并且 *b* 是不能被任何质数的平方整除的正整数。问$a+b+c$是多少？

$\textbf{(A)}\ 110 \qquad \textbf{(B)}\ 114 \qquad \textbf{(C)}\ 118 \qquad \textbf{(D)}\ 122 \qquad \textbf{(E)}\ 126$

Problem 25

Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2,$ and $d(12) = 6$. (This function is known as the divisor function.)

Let $f(n) = \frac{d(n)}{\sqrt[3]{n}}.$ There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N?

令$d(n)$表示 n 的正约数（包括 1 和 n）的个数。例如，$d(1) = 1, d(2) = 2, d(12) = 6$（这个函数被称为约数函数。）令$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$ 存在唯一的正整数 N，使得对于所有正整数 $n \neq N$，$f(N) > f(n)$。问 N 的各位数字之和是多少？

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9