2021Spring AMC12B

Problem 1

How many integer values of x satisfy $|x| < 3\pi$?

有多少个整数值满足$|x| < 3\pi$?

$\textbf{(A)}\ 9 \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 18 \qquad \textbf{(D)}\ 19 \qquad \textbf{(E)}\ 20$

Problem 2

At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?

在一次数学竞赛中，57 名学生穿着蓝色衬衫，另外 75 名学生穿着黄色衬衫。132 名学生被分成了 66 对。这其中恰好有 23 对，每对的两名学生都穿着蓝色衬衫。问两名学生都穿着黄色衬衫的对有多少个？

$\textbf{(A)}\ 23 \qquad \textbf{(B)}\ 32 \qquad \textbf{(C)}\ 37 \qquad \textbf{(D)}\ 41 \qquad \textbf{(E)}\ 64$

Problem 3

Suppose $$2+\frac{1}{1+\frac{1}{2+\frac{2}{3+x}}}=\frac{144}{53}.$$ What is the value of x?

$$2+\frac{1}{1+\frac{1}{2+\frac{2}{3+x}}}=\frac{144}{53}.$$ 求 x 的值。

$\textbf{(A)}\ \frac{3}{4} \qquad \textbf{(B)}\ \frac{7}{8} \qquad \textbf{(C)}\ \frac{14}{15} \qquad \textbf{(D)}\ \frac{37}{38} \qquad \textbf{(E)}\ \frac{52}{53}$

Problem 4

Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the score of all the students?

Blackwell 女士在两个班进行考试。上午班学生的平均分是 84，而下午班的平均分是 70。上午班学生人数与下午班学生人数之比是$\frac{3}{4}$。问所有学生的平均分是多少？

$\textbf{(A)}\ 74 \qquad\textbf{(B)}\ 75 \qquad\textbf{(C)}\ 76 \qquad\textbf{(D)}\ 77 \qquad\textbf{(E)}\ 78$

Problem 5

The point $P(a,b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1,5)$ and then reflected about the line $y = -x$. The image of P after these two transformations is at $(-6,3)$. What is $b - a$?

xy 坐标平面中的点$P(a,b)$首先绕着点（1，5）逆时针旋转 90°，然后沿直线$y=-x$反射。

经过这两次变换后 *P* 的影像是点（−6，3）。问$b-a$是多少？

$\textbf{(A)}\ 1 \qquad\textbf{(B)}\ 3 \qquad\textbf{(C)}\ 5 \qquad\textbf{(D)}\ 7 \qquad\textbf{(E)}\ 9$

Problem 6

An inverted cone with base radius $12 \mathrm{cm}$ and height $18 \mathrm{cm}$ is full of water. The water is poured into a tall cylinder whose horizontal base has a radius of $24 \mathrm{cm}$. What is the height in centimeters of the water in the cylinder?

一个底部半径为 12 厘米，高度为 18 厘米的倒置圆锥体中充满了水。将水倒入一个水平底部半径为 24 厘米的高圆柱体中。问圆柱体中水的高度是多少厘米？

$\textbf{(A)}\ 1.5 \qquad\textbf{(B)}\ 3 \qquad\textbf{(C)}\ 4 \qquad\textbf{(D)}\ 4.5 \qquad\textbf{(E)}\ 6$

Problem 7

Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

令 N=34・34・63・270。N 的奇约数之和与 N 的偶约数之和的比值是多少？

(A) $1:16$ **(B)** $1:15$ **(C)** $1:14$ **(D)** $1:8$ **(E)** $1:3$

Problem 8

Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38,$ and 34. What is the distance between two adjacent parallel lines?

三条等间距的平行线与一个圆相交，形成三条长度分别为 38，38 和 34 的弦。问相邻的两条平行线之间的距离是多少？

(A) $5\frac{1}{2}$ **(B)** 6 **(C)** $6\frac{1}{2}$ **(D)** 7 **(E)** $7\frac{1}{2}$

Problem 9

What is the value of $\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}$?

下列式子的值是多少

$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}?$$

(A) 0 **(B)** 1 **(C)** $\frac{5}{4}$ **(D)** 2 **(E)** $\log_2 5$

Problem 10

Two distinct numbers are selected from the set $\{1, 2, 3, 4, \dots, 36, 37\}$ so that the sum of the remaining 35 numbers is the product of these two numbers. What is the difference of these two numbers?

从集合｛1，2，3，4，…，36，37｝中选出两个不同的数，使得剩下 35 个数的和是选出的两个数的乘积。问这两个数的差是多少？

(A) 5 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Problem 11

Triangle ABC has $AB=13, BC=14$ and $AC=15$. Let P be the point on $\overline{AC}$ such that $PC=10$. There are exactly two points D and E on line BP such that quadrilaterals $ABCD$ and $ABCE$ are trapezoids. What is the distance DE?

在三角形 *ABC* 中，*AB*＝13，*BC*=14，*AC*＝15。设 *P* 是$\overline{AC}$上一点，满足 *PC*＝10。在直线 *BP* 上恰好有两点 *D* 和 *E*，使得四边形 *ABCD* 和 *ABCE* 是梯形。问 *DE* 的长度是多少？

$\textbf{(A)}\ \frac{42}{5} \qquad \textbf{(B)}\ 6\sqrt{2} \qquad \textbf{(C)}\ \frac{84}{5} \qquad \textbf{(D)}\ 12\sqrt{2} \qquad \textbf{(E)}\ 18$

Problem 12

Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

假设 *S* 是正整数的有限集合。如果把 *S* 中最大的整数从 *S* 中移除，则其余的整数的（算术）平均值为 32。如果把 *S* 中的最小整数也移除，则其余的整数的平均值为 35。如果又把最大的整数加回到集合中，则整数的平均值上升到 40。原来集合 *S* 中最大的整数比 *S* 中最小的整数大 72。问集合 *S* 中所有整数的平均值是多少

$\textbf{(A)}\ 36.2 \qquad \textbf{(B)}\ 36.4 \qquad \textbf{(C)}\ 36.6 \qquad \textbf{(D)}\ 36.8 \qquad \textbf{(E)}\ 37$

Problem 13

How many values of θ in the interval $0<\theta\le 2\pi$ satisfy$1-3\sin\theta+5\cos 3\theta=0$?

在区间$0<\theta\le 2\pi$中有多少个θ值，满足$1-3\sin\theta+5\cos 3\theta=0$?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 8$

Problem 14

Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}, \overline{MC},$ and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

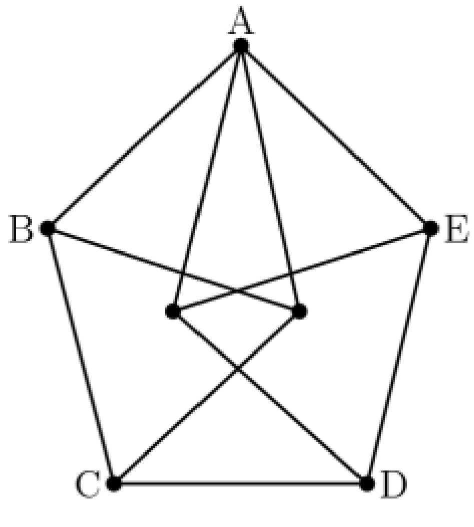
ABCD 是一个矩形，而$\overline{DM}$是垂直于平面 *ABCD* 的线段。假设$\overline{DM}$的长度为整数，并且$\overline{MA}, \overline{MC},$和$\overline{MB}$的长度是连续的正奇数（依此顺序）。问棱锥 *MABCD* 的体积是多少？

$\textbf{(A) } 24\sqrt5 \qquad \textbf{(B) } 60 \qquad \textbf{(C) } 28\sqrt5 \qquad \textbf{(D) } 66 \qquad \textbf{(E) } 8\sqrt{70}$

Problem 15

The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n?$

下图由 11 条线段构成，每条线段的长度都是 2。五边形 *ABCDE* 的面积可以写成 $\sqrt{m} + \sqrt{n}$，其中 *m* 和 *n* 是正整数。问 *m*+*n* 是多少？



$\textbf{(A) } 20 \qquad \textbf{(B) } 21 \qquad \textbf{(C) } 22 \qquad \textbf{(D) } 23 \qquad \textbf{(E) } 24$

Problem 16

Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x)=x^3+ax^2+bx+c$, where $1<a<b<c$. What is $g(1)$ in terms of a, b, and c?

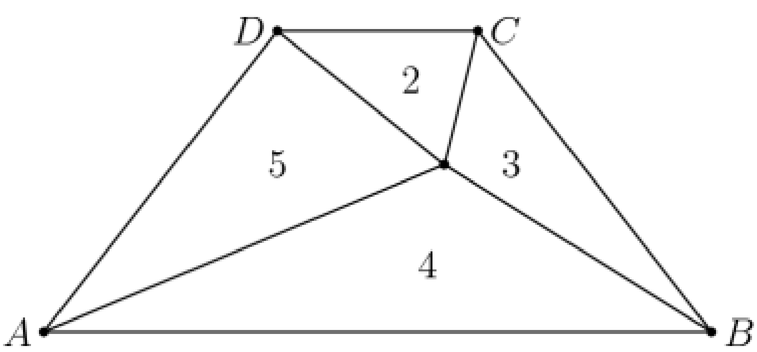
设$g(x)$是一个首项系数为 1 的多项式，它的三个根是$f(x)=x^3+ax^2+bx+c$,的三个根的倒数，其中$1<a<b<c$.问$g(1)$如何用 *a*，*b* 和 *c* 来表示？

(A) $\frac{1+a+b+c}{c}$ **(B)** $1+a+b+c$ **(C)** $\frac{1+a+b+c}{c^2}$ **(D)** $\frac{a+b+c}{c^2}$ **(E)** $\frac{1+a+b+c}{a+b+c}$

Problem 17

Let $ABCD$ be an isosceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB>CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2, 3, 4,$ and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?

在 *ABCD* 等腰梯形中，$\overline{AB}$与$\overline{CD}$是平行的底边，并且 *AB*>*CD*。从 *ABCD* 内一点到其各顶点的线段将梯形分成了四个三角形，从底边为$\overline{CD}$的三角形开始，沿顺时针方向，面积分别为 2，3，4 和 5，如下图所示。求$\frac{AB}{CD}$?



(A) 3 **(B)** $2+\sqrt{2}$ **(C)** $1+\sqrt{6}$ **(D)** $2\sqrt{3}$ **(E)** $3\sqrt{2}$

Problem 18

Let z be a complex number satisfying $12|z|^2=2|z+2|^2+|z^2+1|^2+31.$ What is the value of $z+\frac{6}{z}$?

假设复数 z 满足$12|z|^2=2|z+2|^2+|z^2+1|^2+31.$ 那么$z+\frac{6}{z}$?的值是多少?

$\textbf{(A)}\ -2 \qquad \textbf{(B)}\ -1 \qquad \textbf{(C)}\ \frac{1}{2} \qquad \textbf{(D)}\ 1 \qquad \textbf{(E)}\ 4$

Problem 19

Two fair dice, each with at least 6 faces are rolled. On each face of each dice is printed a distinct integer from 1 to the number of faces on that die, inclusive. The probability of rolling a sum if 7 is $\frac{3}{4}$ of the probability of rolling a sum of $10,$ and the probability of rolling a sum of 12 is $\frac{1}{12}$. What is the least possible number of faces on the two dice combined?

抛掷两个均匀的骰子，每个骰子都至少有 6 个面。就每个骰子而言，每个面上都印有一个不同的整数，取值分别是从 1 开始到该骰子面数之间的那些整数。两个骰子掷出的数总和为 7 的概率是掷出的数总和为 10 的概率的$\frac{3}{4}$，并且掷出的数总和为 12 的概率是$\frac{1}{12}$。问两个骰子面数之和的最小可能值是多少？

$\textbf{(A)}\ 16 \qquad \textbf{(B)}\ 17 \qquad \textbf{(C)}\ 18 \qquad \textbf{(D)}\ 19 \qquad \textbf{(E)}\ 20$

Problem 20

Let $Q(z)$ and $R(z)$ be the unique polynomials such that $z^{2021}+1=(z^2+z+1)Q(z)+R(z)$ and the degree of R is less than $2.$ What is $R(z)$?

设$Q(z)$和$R(z)$是满足$z^{2021}+1=(z^2+z+1)Q(z)+R(z)$以及 *R* 的次数小于 2 的唯一一组多项式。问$R(z)$是多少？

$\textbf{(A)}\ -z \qquad \textbf{(B)}\ -1 \qquad \textbf{(C)}\ 2021 \qquad \textbf{(D)}\ z+1 \qquad \textbf{(E)}\ 2z+1$

Problem 21

Let S be the sum of all positive real numbers x for which $x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}$. Which of the following statements is true?

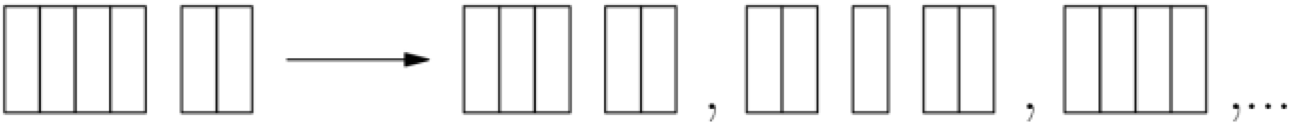
设 S 是所有满足 $x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}$ 的正实数 x 的和。问以下哪个论断正确？

$\textbf{(A)}\ S<\sqrt2 \qquad \textbf{(B)}\ S=\sqrt2 \qquad \textbf{(C)}\ \sqrt2<S<2 \qquad \textbf{(D)}\ 2\le S<6 \qquad \textbf{(E)}\ S\ge 6$

Problem 22

Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3,2), (2,1,2), (4), (4,1), (2,2),$ or $(1,1,2)$.

Arjun 和 Beth 玩一个游戏，他们轮流从一个由砖块组成，可能包括空隙的“墙”上移除一块砖或两块相邻的砖。这些墙的高度都和砖的高度一样。例如，一个由 4 块砖和 2 块砖组成的墙可以通过一次操作变为以下的一种构型：（3，2），（2，1，2），（4），（4，1），（2，2）或者（1，1，2）。



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

Arjun 首先开始，谁取走最后一块砖谁将获胜。对于哪种起始的构型，Beth 可以有必胜策略？

$\textbf{(A)}\ (6,1,1) \qquad \textbf{(B)}\ (6,2,1) \qquad \textbf{(C)}\ (6,2,2) \qquad \textbf{(D)}\ (6,3,1) \qquad \textbf{(E)}\ (6,3,2)$

Problem 23

Three balls are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin i is 2^{-i} for $i = 1, 2, 3, \ldots$. More than one ball is allowed in each bin. The probability that the balls end up evenly spaced in distinct bins is $\frac{p}{q}$, where p and q are relatively prime positive integers. (For example, the balls are evenly spaced if they are tossed into bins $3, 17,$ and 10.) What is $p+q$?

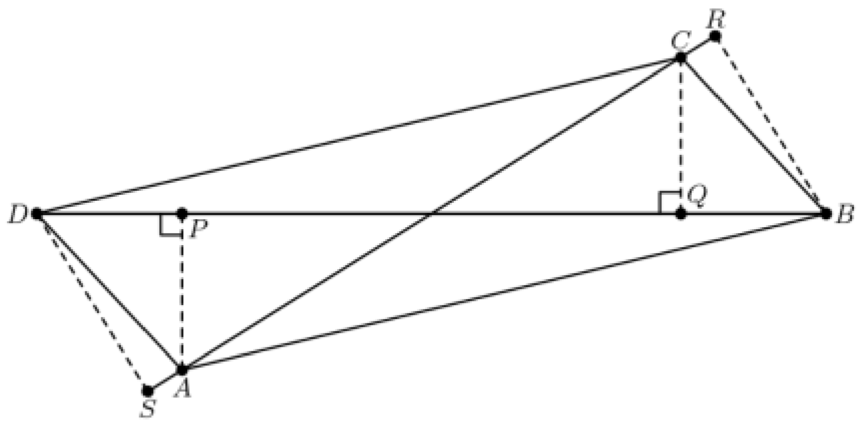
将三个球随机并且相互独立地扔进用正整数编号的桶里，对于每个球而言，它被扔进编号为 i 的桶的概率是2^{-i}， i=1，2，3，...。每个桶里可以有多于一个球。设三个球在不同的桶中并且桶的编号等距分布的概率为 $\frac{p}{q}$，其中 p 和 q 是互质的正整数。（例如，如果球被扔进编号 3，17 和 10 的桶中，那么就认为是等距的。）问 $p+q$ 是多少？

(A) 55　**(B)** 56　**(C)** 57　**(D)** 58　**(E)** 59

Problem 24

Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and $C,$ respectively, onto the line $BD;$ and points R and S are the projections of B and $D,$ respectively, onto the line $AC.$ See the figure, which also shows the relative locations of these points.Suppose $PQ = 6$ and $RS = 8,$ and let d denote the length of $\overline{BD},$ the longer diagonal of $ABCD.$ Then d^2 can be written in the form $m+n\sqrt{p},$ where $m,n,$ and p are positive integers and p is not divisible by the square of any prime. What is $m+n+p?$

ABCD 是面积为 15 的平行四边形。点 *P* 和 *Q* 分别是 *A* 和 *C* 在直线 BD 上的投影；点 *R* 和 *S* 分别是 *B* 和 *D* 在直线 *AC* 上的投影。如图所示，图中也显示了这些点的相对位置关系。假设 *PQ*=6，并且 *RS*=8，用 *d* 表示 *ABCD* 的较长对角线$\overline{BD}$的长度。那么d^2可以写成$m+n\sqrt{p}$的形式，其中 *m*，*n* 和 *p* 是正整数，*p* 不被任何质数的平方所整除。问 *m* + *n* + *p* 的值是什么？



$\textbf{(A) }81 \qquad \textbf{(B) }89 \qquad \textbf{(C) }97 \qquad \textbf{(D) }105 \qquad \textbf{(E) }113$

Problem 25

Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and $30,$ inclusive. Exactly 300 points in S lie on or below a line with equation $y=mx.$ The possible values of m lie in an interval of length $\frac ab,$ where a and b are relatively prime positive integers. What is $a+b?$

设 *S* 是坐标平面上横纵坐标都是从 1 到 30 之间（包括 1 和 30）的整数的格点组成的集合。在 *S* 中恰好有 300 个点在解析式是$y=mx.$的直线上或者位于该直线的下方。*m* 的可能值构成长度为 $\frac{a}{b}$ 的区间，其中 *a* 和 *b* 是互质的正整数。问 *a* + *b* 是多少？

$\textbf{(A) }31 \qquad \textbf{(B) }47 \qquad \textbf{(C) }62 \qquad \textbf{(D) }72 \qquad \textbf{(E) }85$