2025 Euclid 数学竞赛 · 答案解析

手写解答文字转录版

第 1 题

> **(a)** If $4(x-2)=2(x-4)$, what is the value of x?
>
> **(b)** If $2x=9$, what is the value of 2^{6x-23}?
>
> **(c)** Determine the coordinates of the point of intersection of the lines with equations $y=3x+7$ and $y=7x+3$.

(a) 两边同除以 2：

$$2(x-2)=x-4 \;\Rightarrow\; 2x-4=x-4 \;\Rightarrow\; \boldsymbol{x=0}$$

(b) 由 $2x=9$：

$$2^{6x-23}=2^{3\cdot 2x-23}=2^{3\cdot 9-23}=2^{4}=\boldsymbol{16}$$

(c) 联立两式：

$$\begin{cases} y=3x+7 \\ y=7x+3 \end{cases} \Rightarrow\; 3x+7=7x+3 \;\Rightarrow\; 4x=4 \;\Rightarrow\; x=1,\; y=10$$

交点坐标为 $(1,\,10)$。

第 2 题

> **(a)** There is one positive integer k for which $3<\sqrt{k^2+4}<4$. What is this positive integer k?
>
> **(b)** What is the sum of the 20 smallest odd positive integers?
>
> **(c)** In the diagram, $\triangle BDF$ is equilateral and $\angle FAB=\angle BCD=\angle DEF=90^\circ$. Also, $AB=16$, $BC=8$, $DE=5$, and $FA=13$. Determine the perimeter of hexagon $ABCDEF$.

(a) 不等式各项平方：

$$9<k^2+4<16 \;\Rightarrow\; 5<k^2<12 \;\Rightarrow\; k^2=9 \;\Rightarrow\; \boldsymbol{k=3} \quad (\text{since } k>0)$$

(b) 最小的 20 个正奇数为 $1,3,\ldots,39$：

$$1+3+\cdots+39=\frac{1+39}{2}\cdot 20=\boldsymbol{400}$$

(c) 在直角三角形 FAB 中（$\triangle BDF$ 为等边三角形）：

$$BF^2=AB^2+AF^2=16^2+13^2=425=BD^2=DF^2$$

$$CD=\sqrt{BD^2-BC^2}=\sqrt{425-8^2}=19,\qquad EF=\sqrt{DF^2-DE^2}=\sqrt{425-25}=$$

$$\text{perimeter}=16+8+19+5+20+13=\mathbf{81}$$

第 3 题

> **(a)** Suppose that $p+q+r=18$ and $p+q=5$ and $q+r=9$. What is the value of q?
>
> **(b)** The line with equation $6x+y=24$ has its x-intercept at point P and its y-intercept at point Q. What is an equation of the parabola whose y-intercept is at Q and whose only x-intercept is at P?
>
> **(c)** Suppose that $\dfrac{1}{2w}=\dfrac{1}{3y}=\dfrac{1}{4z}$ and $\dfrac{1}{2w}+\dfrac{1}{3y}+\dfrac{1}{4z}=\dfrac{1}{24}$. Determine the value of $w+y+z$.

(a) 记三式为 ①②③，则 ②+③−①：

$$q=5+9-18=\mathbf{-4}$$

(b) 令 $x=0$ 得 $y=24$，即 $Q(0,24)$；令 $y=0$ 得 $6x=24$，$x=4$，即 $P(4,0)$。

抛物线只有唯一 x 截距 P，设 $y=k(x-4)^2=k(x^2-8x+16)$。代入 $Q(0,24)$：

$$k(0-0+16)=24\ \Rightarrow\ k=\frac{3}{2}\ \Rightarrow\ \boldsymbol{y=\tfrac{3}{2}(x-4)^2}$$

(c) 由等比性质：

$$\frac{1}{2w}=\frac{1}{3y}=\frac{1}{4z}=\frac{\frac{1}{24}}{3}=\frac{1}{72}\ \Rightarrow\ w=36,\ y=24,\ z=18$$

$$w+y+z=36+24+18=\mathbf{78}$$

第 4 题

> **(a)** Terry's bicycle has a larger front wheel with radius 15 cm and a smaller rear wheel with radius 9 cm, as shown. Terry ties a ribbon to the top of each wheel, and then starts to ride forward. Terry travels d cm forward and stops. Both ribbons are again at the top of the wheels. What is the integer closest to the smallest possible value of d with $d>0$?
>
> **(b)** In the diagram, $ABCD$ is a rectangle with $AB=24$ and $AD=18$. Also, E is on BC with $EC=6$. If segments DE and AC intersect at F, determine the length of CF.

(a) 设前轮转过 m 圈、后轮转过 n 圈，则

$$d=m\cdot 2\pi\cdot 15=n\cdot 2\pi\cdot 9\ \Rightarrow\ 15m=9n$$

由 $\operatorname{lcm}(15,9)=45$，得 $m_{\min}=3$（此时 $n_{\min}=5$），于是

$$d_{\min}=3\cdot 2\pi\cdot 15=90\pi\approx\mathbf{283}\ \text{cm}$$

(b) 由 $AD \parallel CE$ 得 $\triangle ADF \sim \triangle CEF$，故

$$\frac{AD}{CE} = \frac{AF}{CF} = \frac{18}{6} = \frac{3}{1} \Rightarrow \frac{AC}{CF} = \frac{AF+CF}{CF} = \frac{3+1}{1} = \frac{4}{1}$$

在 $\mathrm{Rt}\triangle ADC$ 中：

$$AC = \sqrt{AD^2 + DC^2} = \sqrt{18^2 + 24^2} = 30 = AF + CF = 4CF \Rightarrow CF = \frac{1}{4}AC = \frac{1}{4}$$

第 5 题

(a) Alice has a lock whose combination consists of three integers a, b, c which need to be entered in that order. The three integers satisfy the following:

• each of a, b and c is between 1 and 40, inclusive;
• a, b and c are all different;
• b is less than a, and b is less than c; and
• one of the integers is 20 and another of the integers is 30.

How many possible combinations satisfy these conditions?

(b) For some angles θ, the three numbers $2 - 2\cos\theta$, $1 + \sin\theta$, $2 + 2\cos\theta$ form a geometric sequence in that order. Determine all possible exact values of $\cos\theta$.

(a) 分类讨论：

情形 1： $b < 20$，则 $\{a, c\} = \{20, 30\}$，有 2 种排法；b 可取 $1 \sim 19$ 共 19 种：$1 \times 2 \times 19 = 38$ 种。

情形 2： $b = 20$，则 $a = 30$ 或 $c = 30$（2 种），剩余一数取 $21 \sim 40$ 中除 30 外的 19 个数：$19 \times 2 = 38$ 种。

合计 $38 + 38 = \mathbf{76}$ 种。

(b) 由等比数列性质（注意 $\sin^2\theta + \cos^2\theta = 1$）：

$$(1+\sin\theta)^2 = (2-2\cos\theta)(2+2\cos\theta)$$

$$1 + 2\sin\theta + \sin^2\theta = 4 - 4\cos^2\theta = 4 - 4(1-\sin^2\theta) = 4\sin^2\theta$$

$$3\sin^2\theta - 2\sin\theta - 1 = 0 \Rightarrow (3\sin\theta + 1)(\sin\theta - 1) = 0 \Rightarrow \sin\theta = -\frac{1}{3} \text{ or } 1$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\frac{\mathbf{2\sqrt{2}}}{\mathbf{3}} \text{ or } \mathbf{0}$$

第 6 题

> **(a)** Twelve points are equally spaced around the entire circumference of a circle. In how many ways can three of these points be chosen so that the triangle that they form has at least two sides of equal length?
>
> **(b)** In the diagram, quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$, $AB = 56$, $BC = 33$, $CD = 39$, and $DA = 52$. Point P is on AD so that BP is perpendicular to AD. Determine the exact length of BP.

(a) 以每个点为顶点的等腰（非等边计数口径）：$12 \times 5 = 60$；等边三角形共 4 个，每个被 3 个顶点各算一次：$4 \times 3 = 12$。故总数为

$$60 - 12 + 4 = \mathbf{52}$$

(b) 设 $PD = x$，则 $AP = 52 - x$。过 C 作 $CE \perp BP$ 于 E，则 $CEPD$ 为矩形，$EC = x$，$EP = CD = 39$。由 $\triangle BCE \sim \triangle ABP$：

$$\frac{BC}{AB} = \frac{BE}{AP} \;\Rightarrow\; \frac{33}{56} = \frac{\sqrt{33^2 - x^2}}{52 - x}$$

解得 $x = \dfrac{396}{13}$，于是 $BE = \dfrac{165}{13}$，

$$BP = BE + EP = \frac{165}{13} + 39 = \boxed{\mathbf{\frac{672}{13}}}$$

第 7 题

> **(a)** The functions f and g are defined by the tables of values below:
>
> $$f\text{: } (1,5),(2,3),(3,4),(4,1),(5,2) \quad | \quad g\text{: } (1,3),(2,1),(3,4),(4,5),(5,2)$$
>
> The functions f^{-1} and g^{-1} are the inverse functions of f and g, respectively. If $f^{-1}(g^{-1}(a)) = 3$, what is the value of a?
>
> **(b)** Determine all pairs (x, y) of real numbers that satisfy the following system of equations:
>
> $$x^2 - 8xy + 16y^2 = 0$$
> $$(\log_{10} x)^2 + 2(\log_{10} x)(\log_{10} y) + (\log_{10} y)^2 = 4$$

(a)

$$f^{-1}(g^{-1}(a)) = 3 \;\Rightarrow\; f\big[f^{-1}(g^{-1}(a))\big] = f(3) \;\Rightarrow\; g^{-1}(a) = f(3) = 4$$
$$\Rightarrow\; g\big[g^{-1}(a)\big] = g(4) \;\Rightarrow\; a = g(4) = \mathbf{5}$$

(b) 定义域：$x > 0,\ y > 0$。第一式：

$$(x - 4y)^2 = 0 \;\Rightarrow\; x = 4y$$

第二式：

$$(\log_{10}x+\log_{10}y)^2=4\ \Rightarrow\ (\log_{10}xy)^2=4\ \Rightarrow\ \log_{10}xy=\pm2$$

代入 $x=4y$：$\log_{10}4y^2=\pm2\ \Rightarrow\ 4y^2=100$ or $\dfrac{1}{100}\ \Rightarrow\ y^2=25$ or $\dfrac{1}{400}$，故 $y=5$ or $\dfrac{1}{20}$，对应 $x=20$ or $\dfrac{1}{5}$。

$$(x,y)=\mathbf{(20,\ 5)}\text{ or }\left(\tfrac{1}{5},\ \tfrac{1}{20}\right)$$

第 8 题

> **(a)** Leilei, Jerome and Farzad write a test independently. The probability that Leilei passes the test and Jerome fails the test is $\frac{3}{20}$. The probability that Jerome passes and Farzad fails is $\frac{1}{4}$. The probability that Leilei and Farzad both pass is $\frac{2}{5}$. Determine the probability that at least one of Leilei, Jerome and Farzad fails the test.
>
> **(b)** The integer 7447 is a palindrome because it reads the same forwards and backwards. Suppose that positive integers m and n are palindromes between 1001 and 9999, inclusive, with $m>n$. Determine the number of pairs (m,n) for which the difference $m-n$ is a multiple of 35.

(a) 记三人通过的概率分别为 L,J,F：

$$L(1-J)=\frac{3}{20}\quad①,\qquad J(1-F)=\frac{1}{4}\quad②,\qquad LF=\frac{2}{5}\quad③$$

①÷③：$\dfrac{1-J}{F}=\dfrac{3}{8}$，即 $8(1-J)=3F$。由 ② 得 $J=\dfrac{1}{4(1-F)}$，代入：

$$8(1-F)-2=3F(1-F)\ \Rightarrow\ 8-8F-2=3F-3F^2\ \Rightarrow\ 3F^2-11F+6=0$$

$$(3F-2)(F-3)=0\ \Rightarrow\ F=\frac{2}{3},\quad L=\frac{3}{5},\quad J=\frac{3}{4}$$

至少一人不及格的概率：

$$P=1-\frac{3}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}=\mathbf{\frac{7}{10}}$$

(b) 四位回文数 $ABBA=A\cdot10^3+B\cdot10^2+B\cdot10^1+A\cdot10^0=1001A+110B$，其中 $A\in\{1,2,\dots,9\}$，$B\in\{0,1,\dots,9\}$。

$$35\mid(m-n)\ \Rightarrow\ 5\mid(m-n)\text{ and }7\mid(m-n)$$

设 $m=A_1B_1B_1A_1$，$n=A_2B_2B_2A_2$，则

$$m-n=\underbrace{1001(A_1-A_2)}_{7\,|\,1001}+\underbrace{110(B_1-B_2)}_{5\,|\,110}\ \Rightarrow\ 5\mid(A_1-A_2)\text{ and }7\mid(B_1-B_2)$$

即 $A_1-A_2\in\{0,\pm5\}$，$B_1-B_2\in\{0,\pm7\}$（保证 $m>n$）：

情形 1：$A_1 = A_2$（9 种），需 $B_1 > B_2$：$(B_1, B_2) = (7,0), (8,1), (9,2)$ 共 3 种 $\Rightarrow 9 \times 3 = 27$。

情形 2：$A_1 > A_2$：$(A_1, A_2) = (6,1), (7,2), (8,3), (9,4)$ 共 4 种；此时 $m > n$ 恒成立，B 可取 $B_1 = B_2$（10 种）或 $|B_1 - B_2| = 7$（3×2 种）$\Rightarrow 4 \times (10 + 3 \times 2) = 64$。

（$A_1 < A_2$ 时 $m < n$，不合要求。）合计 $27 + 64 = \mathbf{91}$ 对。

第 9 题

> Suppose that $p(x) = qx^3 - rx^2 - sx + t$ for some positive integers $q < r < s < t$ which form an arithmetic sequence.
>
> **(a)** Show that $x = 1$ is a root of $p(x)$.
>
> **(b)** Suppose that the average of q, r, s, t is 19 and that $p(x)$ has three rational roots. Determine the roots of $p(x)$.
>
> **(c)** Prove that, for every positive integer $n > 3$, there are at least two arithmetic sequences of positive integers $q < r < s < t$ with common difference $2n$ for which $p(x)$ has three rational roots.

(a) $p(1) = q - r - s + t = (q + t) - (r + s)$。等差数列性质给出 $r + s = q + t$，故 $p(1) = 0$，即 $x = 1$ 是 $p(x)$ 的根。

(b) 由 (a) 设 $p(x) = (x-1)(Ax^2 + Bx + C)$，展开比较系数：

$$Ax^3 + (B - A)x^2 + (C - B)x - C = qx^3 - rx^2 - sx + t \;\Rightarrow\; A = q,\ B = q - r,\ C =$$

$$p(x) = (x-1)\big[qx^2 + (q - r)x - t\big]$$

由题设平均数为 19，记 $q = 19 - 3m$，$r = 19 - m$，$s = 19 + m$，$t = 19 + 3m$，则

$$p(x) = (19 - 3m)x^3 - (19 - m)x^2 - (19 + m)x + (19 + 3m) = (x-1)\big[(19 - 3m)x^2$$

$p(x)$ 有 3 个有理根，即二次因子有 2 个有理根，判别式须为完全平方数：

$$\Delta = (-2m)^2 + 4(19 - 3m)(19 + 3m) = 4m^2 + 4(19^2 - 9m^2) = 1444 - 32m^2 = 4(36$$

又 $19 - 3m > 0$ 且 m 为正整数，则 $m = 1, 2, 3, 4, 5, 6$。经验证仅 $m = 3$ 符合，此时 $\Delta = 4 \times 17^2$，

$$p(x) = (x-1)(10x^2 - 6x - 28) = 2(x-1)(5x^2 - 3x - 7) = 2(x-1)(5x - 7)(x + 2)$$

三个有理根为 $\mathbf{1},\ \mathbf{\dfrac{7}{5}},\ \mathbf{-2}$。

(c) 公差为 $2n$，设 $q = a$，$r = a + 2n$，$s = a + 4n$，$t = a + 6n$（$a > 0$），则

$$p(x) = (x-1)\big[ax^2 - 2nx - (a + 6n)\big]$$

判别式须为完全平方数 $(2m)^2$：

$$\Delta = 4n^2 + 4a(a + 6n) = 4(a^2 + 6an + 9n^2) - 32n^2 = (2m)^2 \;\Rightarrow\; (a + 3n)^2 - m^2 =$$

$$(a+3n+m)(a+3n-m)=8n^2$$

两因子同偶。若取 $a+3n+m=4n$，$a+3n-m=2n$，则 $m=n$，$(a+3n)^2=9n^2$，$a=0$ 或 $-6n$，与 $a>0$ 矛盾，舍去。取以下两种分解：

分解 1：$a+3n-m=2$，$a+3n+m=4n^2 \Rightarrow m=2n^2-1$，

$$(a+3n)^2=(2n^2-1)^2+8n^2=4n^4+4n^2+1=(2n^2+1)^2 \Rightarrow a_1=2n^2-3n+1$$

分解 2：$a+3n-m=4$，$a+3n+m=2n^2 \Rightarrow m=n^2-2$，

$$(a+3n)^2=(n^2-2)^2+8n^2=n^4+4n^2+4=(n^2+2)^2 \Rightarrow a_2=n^2-3n+2$$

比较两者：

$a_1-a_2=n^2-1>0 \quad (n>3)$，　　$a_2=n^2-3n+2=(n-2)(n-1)>0 \quad (n>3)$

故 $a_1>a_2>0$，即得到两个不同的正整数等差数列 $a_1,\ a_1+2n,\ a_1+4n,\ a_1+6n$ 与 $a_2,\ a_2+2n,\ a_2+4n,\ a_2+6n$，均使 $p(x)$ 有三个有理根。证毕。

第 10 题

> An equilateral triangle is formed using n rows of coins. There is 1 coin in the first row, 2 coins in the second row, 3 coins in the third row, and so on, up to n coins in the nth row. Initially, all of the coins show heads (H). Carley plays a game in which, on each turn, she chooses three mutually adjacent coins and flips these three coins over. To win the game, all of the coins must be showing tails (T) after a sequence of turns. An example game with 4 rows of coins after a sequence of two turns is shown.
>
> *Note:* For a triangle with 4 rows of coins, there are 9 possibilities for the set of three coins that Carley can flip on a given turn, denoted $T(1,1), T(2,1), T(2,2), T(2,3), T(3,1), T(3,2), T(3,3), T(3,4), T(3,5)$.
>
> **(a)** If there are 3 rows of coins, give a sequence of 4 turns that results in a win.
>
> **(b)** Suppose that there are 4 rows of coins. Determine whether or not there is a sequence of turns that results in a win.
>
> **(c)** Determine all values of n for which it is possible to win the game starting with n rows of coins.

(a) 依次执行以下 4 步即可获胜（每个硬币恰好被翻转奇数次）：

$$T(1,1),\quad T(2,1),\quad T(2,3),\quad T(2,2)$$

(b) 无法获胜。将 10 个硬币编号 $1\sim10$。奇偶性分析：

① 硬币 1、7、10 分别只被 $T(1,1)$、$T(3,1)$、$T(3,5)$ 覆盖，要变成 T，这三个操作各需奇数次；

② 要使硬币 $2,3,4,6,8,9$ 变成 T，则 $T(2,1),T(2,2),T(2,3),T(3,2),T(3,3),T(3,4)$ 各需偶数次；

③ 但硬币 5 恰好被这 6 个操作覆盖，被翻转总次数为偶数，仍显示 H，矛盾。

故不存在获胜的操作序列。

(c) 小情形检验：$n=1$ ✗；$n=2$ ✓；$n=3$ ✓（见 (a)）；$n=4$ ✗（见 (b)）；

$n=5$ ✓，操作序列：$T(1,1),\ T(2,1),\ T(2,2),\ T(2,3),\ T(4,1),\ T(4,4),\ T(4,7)$；

$n=6$ ✓，操作序列：
$(1,1),\ (2,1),\ (2,2),\ (2,3),\ (4,1),\ (5,1),\ (5,2),\ (5,3),\ (4,7),\ (5,7),\ (5,8),\ (5,9),\ (4,4)$
（即在 $n=3$ 解法基础上递归构造下面 3 行）。

一般地，n 行可解当且仅当可在 $n-3$ 行解法上递归添加 3 行；结合 $n\equiv 1\pmod 3$ 时的奇偶性矛盾，结论为：

$$n\equiv 0,\,2 \pmod 3 \quad (n\in\mathbb{Z}^+)$$

即 $n=3k+3$ or $3k+2$（$k=0,1,2,\ldots$），等价地 $n=3k$ or $3k-1$（$k=1,2,3,\ldots$）。