2026 Euclid 数学竞赛 · 答案解析

手写解答文字转录版

第 1 题

> **(a)** What is the integer t for which $\dfrac{2t}{3}+\dfrac{3t}{2}=26$?
>
> **(b)** What is the integer x for which $\dfrac{3+x}{4}=\dfrac{6+x}{8}$?
>
> **(c)** Suppose that $y>0$ and $\sqrt{3^2+4^2+12^2}=\sqrt{3^2+4^2}+\sqrt{y^2}$. Determine the value of y.

(a) $\boldsymbol{t=12}$

(b) $\boldsymbol{x=0}$

(c) $\sqrt{3^2+4^2+12^2}=\sqrt{169}=13$，$\sqrt{3^2+4^2}=\sqrt{25}=5$，故

$$13=5+\sqrt{y^2}\xrightarrow{y>0}\sqrt{y^2}=y\ \Rightarrow\ 13=5+y\ \Rightarrow\ \boldsymbol{y=8}$$

第 2 题

> **(a)** The sum of the digits of the positive integer 2026 is $2+0+2+6=10$. What is the smallest integer $n>2026$ whose digits also have a sum of 10?
>
> **(b)** The product of the digits of the integer 313 is $3\cdot1\cdot3=9$. How many integers between 100 and 999, including 313, have the property that the product of their digits is 9?
>
> **(c)** The sum of x, $3x$ and $4y$ is equal to 48. The average of x and y is equal to $3x$. Determine the ordered pair (x,y).

(a) $\boldsymbol{2035}$

(b) 设三位数为 $\overline{abc}$，则 $a\cdot b\cdot c=9=9\cdot1\cdot1=3\cdot3\cdot1$。

$\{a,b,c\}=\{9,1,1\}\Rightarrow 911,\ 191,\ 119$；$\{a,b,c\}=\{3,3,1\}\Rightarrow 331,\ 313,\ 133$。

共 **6** 个整数。

(c)

$$\begin{cases}x+3x+4y=48\\ \dfrac{x+y}{2}=3x\end{cases}\Rightarrow\begin{cases}4x+4y=48\\ y=5x\end{cases}\Rightarrow\ 4x+20x=48\ \Rightarrow\ x=2,\ y=10$$

$$(x,y)=\boldsymbol{(2,\ 10)}$$

第 3 题

> **(a)** What is the smallest positive integer n for which $\dfrac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{n}$ is equal to k^3 for some integer k?
>
> **(b)** What is the ordered pair (a, b) that satisfies both of the equations $3^{a+b} = 27$ and $a - b = -5$?
>
> **(c)** For some real number c, the parabola with equation $y = -x^2 + 7x + c$ intersects the x-axis at points $P(10, 0)$ and Q. If the parabola intersects the y-axis at R, determine the area of $\triangle PQR$.

(a) 分解质因数：

$$9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3 = 2^{2+1+3}\cdot 3^{1+1+2}\cdot 5^1\cdot 7^1 = 2^6\cdot 3^4\cdot 5^1\cdot 7^1$$

$$n_{\min} = 3^1\cdot 5^1\cdot 7^1 = \boxed{105}$$

(b) $3^{a+b} = 27 \Rightarrow a+b = 3$，又 $a-b=-5$，故 $a=-1,\ b=4$：

$$(a,b) = \boxed{(-1,\ 4)}$$

(c) $P(10,0)$ 在抛物线上 $\Rightarrow 0 = -10^2 + 70 + c \Rightarrow c = 30$。

$$y = -x^2+7x+30 = -(x-10)(x+3) \ \Rightarrow\ Q(-3,0),\ R(0,30)$$

$$[PQR] = \frac{1}{2}\times 13\times 30 = \boxed{195}$$

第 4 题

> **(a)** In January, Rebecca measured the temperature in Yellowknife every day at 11:00 a.m. The average of these 31 temperatures was -20°C. The average of the temperatures from the first 21 days was -15°C. What was the average of the temperatures from the last 10 days?
>
> **(b)** McKayla runs to her grandmother's house and then runs home along the same straight road. The route from McKayla's house, M, to her grandmother's house, G, is on flat ground from M to H, and then uphill from H to G. The distance from M to H to G is 10 km (that is, $MH + HG = 10$ km). McKayla runs on flat ground at 12 km/h, uphill at 10 km/h, and downhill at 15 km/h. It takes 54 minutes for her to run from M to H to G. Determine the number of minutes that it takes for her to run from G to H to M.

(a) $(-20)\times 31 = -620$，$(-15)\times 21 = -315$，故

$$\frac{(-620)-(-315)}{10} = \boxed{-30.5^\circ\text{C}}$$

(b) 设 $MH = x$ km，则 $HG = 10 - x$ km：

$$\frac{x}{12}+\frac{10-x}{10}=\frac{54}{60}\ \Rightarrow\ x=6$$

返程（$G\to H$ 下坡，$H\to M$ 平地）：

$$\frac{10-6}{15}+\frac{6}{12}=\frac{23}{30}\ \text{hr}=\mathbf{46}\ \text{min}$$

第 5 题

(a) Two fair dice, called D_1 and D_2, each have six faces. D_1 has the numbers $1, 2, 3, 4, 5, 6$ on its faces. D_2 has a 1 on some of its faces and a 2 on its remaining faces. When D_1 and D_2 are rolled, the probability that the sum of the numbers on the top faces is a prime number is $\frac{23}{36}$. How many faces on D_2 have the number 1 on them?

(b) In the diagram, square $ABCD$ has A and B on the x-axis and C and D below the x-axis on the parabola with equation $y = x^2 - 4$. Determine the area of $ABCD$, writing your answer in the form $r - \sqrt{t}$ for some positive integers r and t.

(a) 设 D_2 上 1 的面数为 x，则 2 的面数为 $6-x$。可能的质数和为 $2,3,5,7$。按有序对计数（如 $(1,1),(2,1),(4,1),\ldots$）：

$$23=\underbrace{x}_{\text{和}=2}+\underbrace{(x+6-x)}_{\text{和}=3}+\underbrace{(x+6-x)}_{\text{和}=5}+\underbrace{(x+6-x)}_{\text{和}=7}$$

$$23=18+x\ \Rightarrow\ \boldsymbol{x=5}$$

(b) 设 $C=(x,y)$，则 $x>0,\ y<0$。C 在抛物线上 $\Rightarrow y=x^2-4$；$ABCD$ 是正方形 $\Rightarrow |y|=2|x|$：

$$4-x^2=2x\ \Rightarrow\ x^2+2x-4=0\ \Rightarrow\ x=\frac{-2\pm\sqrt{4+16}}{2}=-1\pm\sqrt{5}$$

由 $x>0$ 得 $x=-1+\sqrt{5}$，故

$$[ABCD]=(2x)^2=4(-1+\sqrt{5})^2=4(1+5-2\sqrt{5})=24-8\sqrt{5}=\mathbf{24-\sqrt{320}}$$

第 6 题

(a) Xander, Yasmin and Zhe each have a rope. Xander's rope is 10 m long. Yasmin's rope is $n\%$ longer than Xander's rope. Zhe's rope is $(2n)\%$ longer than Yasmin's rope. Zhe's rope is $(3.14n)\%$ longer than Xander's rope. If $n > 0$, what is the value of n?

(b) In the diagram, quadrilateral $ABCD$ has $AB = AD = 4$. Also, $\angle ABC = 45^\circ$ and $\angle CDA = 135^\circ$. Determine the exact value of $BC - CD$.

(a) $X = 10$，$Y = 10 \cdot (1 + n\%) = 10 \cdot \dfrac{100+n}{100}$，$Z = 10 \cdot \dfrac{100+n}{100} \cdot \dfrac{100+2n}{100}$。又 $Z = X \cdot (1 + (3.14n)\%) = 10 \cdot \dfrac{100+3.14n}{100}$，故

$$10 \cdot \frac{(100+n)(100+2n)}{100^2} = 10 \cdot \frac{100+3.14n}{100} \;\Rightarrow\; \frac{100^2+300n+2n^2}{100^2} = \frac{100+3.14n}{100}$$

$$100^2 + 300n + 2n^2 = 100^2 + 314n \;\Rightarrow\; 2n^2 = 14n \;\Rightarrow\; \boldsymbol{n = 7}$$

(b) 在 $\triangle ABC$ 中由余弦定理：

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos\angle ABC = 16 + BC^2 - 2 \cdot 4 \cdot BC \cdot \frac{\sqrt{2}}{2} = 16$$

在 $\triangle ACD$ 中由余弦定理（$\cos 135^\circ = -\frac{\sqrt{2}}{2}$）：

$$AC^2 = AD^2 + CD^2 - 2 \cdot 4 \cdot CD \cdot \cos\angle ADC = 16 + CD^2 + 4\sqrt{2}\,CD$$

两式相等：

$$16 + BC^2 - 4\sqrt{2}\,BC = 16 + CD^2 + 4\sqrt{2}\,CD \;\Rightarrow\; BC^2 - CD^2 = 4\sqrt{2}\,(BC + CD)$$

$$BC - CD = \boldsymbol{4\sqrt{2}}$$

第 7 题

> **(a)** In a garden, there are roses and carnations, and no other kind of flower. Each flower is either yellow or white. Half of the yellow flowers are roses, $\dfrac{1}{4}$ of the roses are yellow, and $\dfrac{1}{4}$ of all of the flowers are white carnations. What fraction of all of the flowers are yellow roses?
>
> **(b)** The function f is defined for every real number $x > 0$ in the following way:
>
> $$f(x) = \begin{cases} 0 & \text{if } 0 < x \le 1 \\ 1 + f(\log_2 x) & \text{if } x > 1 \end{cases}$$
>
> Determine the number of positive integers n for which $f(n) = 4$.

(a) 列表（设黄玫瑰为 x，白康乃馨为 y）：

	rose	**carnation**
Yellow	x	x
White	$3x$	y

$$y = \frac{1}{4}(x + x + 3x + y) = \frac{5}{4}x + \frac{y}{4} \;\Rightarrow\; \frac{3}{4}y = \frac{5}{4}x \;\Rightarrow\; \frac{x}{y} = \frac{3}{5}$$

$$\Rightarrow\; \text{Yellow rose} = \frac{x}{5x+y} = \frac{3}{5\times 3 + 5} = \boldsymbol{\frac{3}{20}}$$

(b) 设 $f(n) = 4$，$n \ge 1$，$f(1) = 0$，逐层展开：

$$4 = 1 + f(\log_2 n) \;\Rightarrow\; 3 = f(\log_2 n) \;\Rightarrow\; 2 = f(\log_2(\log_2 n)) \;\Rightarrow\; 1 = f(\log_2(\log_2(\log$$

$$\Rightarrow\; 0 = f\left[\log_2(\log_2(\log_2(\log_2 n)))\right] \;\Rightarrow\; 0 < \log_2(\log_2(\log_2(\log_2 n))) \le 1$$

由于 $y = \log_2 x$ 单调递增：

$$1 < \log_2(\log_2(\log_2 n)) \le 2 \;\Rightarrow\; 2 < \log_2(\log_2 n) \le 2^2 = 4 \;\Rightarrow\; 4 = 2^2 < \log_2 n \le 2^4 =$$

$$\Rightarrow\; 16 = 2^4 < n \le 2^{16} = 65536 \;\Rightarrow\; \text{符合条件的整数个数} = \mathbf{65520}$$

第 8 题

> **(a)** In the diagram, the circle has centre O and radius 2. Points A, B, U, and V are on the circle so that AB and UV are diameters. Point C is outside the circle so that AC is tangent to the circle at A and so that BC intersects the circle at U. If $BU = 2UC$, determine the area of $\triangle CUV$.
>
> **(b)** Determine all positive integers k for which there are exactly 100 non-congruent triangles that have positive integer side lengths, one side length equal to k, perimeter equal to $3k$, and one angle with measure greater than 90°.

(a) $AB \perp AC$（切线），$AU \perp BC$（直径所对角）$\Rightarrow \triangle AUC \sim \triangle BUA$，故

$$\frac{AU}{UC} = \frac{BU}{AU} \;\Rightarrow\; AU^2 = UC \cdot BU = 2\,UC^2 \;\Rightarrow\; \sqrt{2} = \frac{AU}{CU} = \frac{BU}{AU}$$

由勾股定理：

$$AB^2 = AU^2 + BU^2 = AU^2 + 2AU^2 = 3AU^2 \;\Rightarrow\; AB = \sqrt{3}\,AU = \sqrt{6}\,CU$$

代入 $AB = 4$：$CU = \dfrac{4\sqrt{6}}{6} = \dfrac{2\sqrt{6}}{3}$，$AU = \dfrac{4\sqrt{3}}{3}$。

UV 为直径 $\Rightarrow VB \perp BC$。又 $\angle VOB = \angle AOU \Rightarrow \triangle VOB \cong \triangle AOU \Rightarrow VB = AU = \dfrac{4\sqrt{3}}{3}$。

$$[CUV] = \frac{1}{2} \cdot \frac{2\sqrt{6}}{3} \cdot \frac{4\sqrt{3}}{3} = \frac{4\sqrt{18}}{9} = \boxed{\frac{4\sqrt{2}}{3}}$$

(b) 不妨设 $\angle A > 90^\circ > \angle C > \angle B$，则 $BC > AB = k > AC$。记 $BC = a$，$AC = b$，则 $a + b = 2k$，即 $b = 2k - a$。

由三角形不等式 $b + k > a \Rightarrow 3k - a > a$，故 $k < a < \frac{3}{2}k$。

由 $\angle A > 90^\circ$：

$$a^2 = k^2 + b^2 - 2kb\cos A > k^2 + b^2 \;\Rightarrow\; a^2 > k^2 + (2k - a)^2$$

$$a^2 > k^2 + 4k^2 - 4ka + a^2 \;\Rightarrow\; 4ka > 5k^2 \;\Rightarrow\; a > \frac{5}{4}k \;\Rightarrow\; \frac{5}{4}k < a < \frac{3}{2}k$$

反之，若 $\frac{5}{4}k < a < \frac{3}{2}k$，则 a，k，$2k - a$ 恰为上述三角形的三边。问题转化为：确定 k 使区间 $\left(\frac{5}{4}k,\ \frac{3}{2}k\right)$ 内恰有 100 个整数。按 $k \bmod 4$ 分类讨论：

情形 1：$k \equiv 0 \pmod 4$：$\dfrac{3}{2}k - \dfrac{5}{4}k - 1 = 100 \Leftrightarrow \boldsymbol{k = 404}$

情形 2：$k \equiv 1 \pmod 4$：$\left\lfloor \dfrac{3k}{2} \right\rfloor - \left\lfloor \dfrac{5k}{4} \right\rfloor = 100 \Leftrightarrow \left\lfloor k + \dfrac{k}{2} \right\rfloor - \left\lfloor k + \dfrac{k}{4} \right\rfloor = 100 \Leftrightarrow \left(k + \dfrac{k-1}{2}\right) - \left(k + \dfrac{k-1}{4}\right) = 100 \Leftrightarrow \boldsymbol{k = 401}$

情形 3：$k \equiv 2 \pmod 4$：$\dfrac{3}{2}k - \left\lfloor \dfrac{5k}{4} \right\rfloor - 1 = 100 \Leftrightarrow \dfrac{3}{2}k - \left(k + \dfrac{k-2}{4}\right) - 1 = 100 \Leftrightarrow \boldsymbol{k = 402}$

情形 4：$k \equiv 3 \pmod 4$：$\left\lfloor \dfrac{3k}{2} \right\rfloor - \left\lfloor \dfrac{5k}{4} \right\rfloor = 100 \Leftrightarrow \left(k + \dfrac{k-1}{2}\right) - \left(k + \dfrac{k-3}{4}\right) = 100 \Leftrightarrow \boldsymbol{k = 399}$

综上，$\boldsymbol{k = 399,\ 401,\ 402,\ 404}$。

第 9 题

> **(a)** Suppose that $\triangle XYZ$ has $XY = XZ$ and $YZ = b$. If the area of $\triangle XYZ$ is 40 and the perimeter of $\triangle XYZ$ is 32, determine a cubic polynomial f with integer coefficients and with the property that $f(b) = 0$.
>
> **(b)** Consider positive real numbers A and P. If there exists an isosceles triangle with area A and perimeter P, prove that there exist at most two non-congruent isosceles triangles with area A and perimeter P.
>
> **(c)** Determine positive integers A and P for which there exist two non-congruent isosceles triangles XYZ with $XY = XZ$, integer side lengths less than 300, area A, and perimeter P, and with the property that the length of YZ in exactly one of the two triangles is a multiple of 7.

(a) 周长为 32 $\Rightarrow XY = XZ = \dfrac{32 - b}{2} = 16 - \dfrac{b}{2}$。由 Heron 公式（半周长为 16）：

$$40^2 = [XYZ]^2 = 16\left[16 - \left(16 - \tfrac{b}{2}\right)\right]^2 (16 - b) \ \Rightarrow\ b^3 - 16b^2 + 400 = 0$$

$$\boldsymbol{f(b) = b^3 - 16b^2 + 400}$$

(b) 设等腰三角形底边长为 b，腰长为 a。则 $P = 2a + b \Rightarrow a = \dfrac{P}{2} - \dfrac{b}{2}$。由 Heron 公式：

$$A^2 = \frac{P}{2}\left(\frac{P}{2} - a\right)^2\left(\frac{P}{2} - b\right) \ \Rightarrow\ 16A^2 = P\,b^2\,(P - 2b)$$

$$\Leftrightarrow\ 2Pb^3 - P^2b^2 + 16A^2 = 0 \qquad (*)$$

由于这样的等腰三角形存在，方程 $(*)$ 有一个正实数解。设 b_1, b_2, b_3 为代数基本定理保证的三个根，其中 $b_1 > 0$。由 Vieta 定理：

$$b_1b_2b_3 = -\frac{8A^2}{P}$$

由于 P、A 为正，b_2b_3 为负，即 b_2, b_3 中一正一负。因此方程 $(*)$ 至多有两个正实数解，即至多存在两个互不全等的等腰三角形满足条件。

(c) 现在 A、P 为正整数。由 (b) 知 $16A^2 = Pb^2(P-2b)$。若 b 为奇数，则 $P = b + 2a$ 也是奇数，从而 $Pb^2(P-2b)$ 为奇数，不可能。故 b 为偶数。

设 b_1, b_2 为 $16A^2 = Pb^2(P-2b)$ 的两个整数解，则

$$Pb_1^2(P-2b_1) = Pb_2^2(P-2b_2) \;\Rightarrow\; P^2(b_1^2-b_2^2) = 2P(b_1^3-b_2^3)$$

$$P(b_1+b_2) = 2(b_1^2+b_1b_2+b_2^2) \qquad (1) \qquad \Rightarrow \qquad P = \frac{2(b_1^2+b_1b_2+b_2^2)}{b_1+b_2}$$

于是

$$16A^2 = b_1^2\,P\,(P-2b_1) = \frac{4b_1^2b_2^2}{(b_1+b_2)^2}\cdot(b_1^2+b_1b_2+b_2^2)$$

故 $b_1^2+b_1b_2+b_2^2 = n^2$ 对某个 $n \in \mathbb{Z}^+$ 成立。这是经典的丢番图方程，可取本原解

$$x = 3^2-1^2 = 8, \qquad y = 2\cdot3\cdot1+1^2 = 7, \qquad n = 3^2+3\cdot1+1^2 = 13$$

设 $b_1 = 8d$，$b_2 = 7d$，则

$$P = \frac{2(b_1^2+b_1b_2+b_2^2)}{b_1+b_2} = \frac{338}{15}d$$

为使 P 为整数，取 $d = 15k$，于是 $b_1 = 120k$，$b_2 = 105k$，$P = 338k$。为使各边长均为整数，需要 $\dfrac{P-b_1}{2} = 109k$ 与 $\dfrac{P-b_2}{2} = \dfrac{233}{2}k$ 均为整数，故 k 为偶数。

$k=2$：$b_1 = 240$，$b_2 = 210$；$k=4$：$b_1 = 480$，$b_2 = 420$，边长超过 300，不符合。

由于 $7 \mid b_2 = 210$（而 $7 \nmid 240$），唯一符合条件的三角形边长为 $(233, 233, 210)$（另一个为 $(218, 218, 240)$）。此时

$$\boldsymbol{P = 676, \quad A = 21840}$$

第 10 题

> Suppose that n is a positive integer with $n \geq 5$. An arrangement $a_1, a_2, \ldots, a_{n-1}, a_n$ of the n positive integers $1, 2, \ldots, n-1, n$ is said to have an *internal peak* at position t (with $2 \leq t \leq n-1$) if $a_{t-1} < a_t$ and $a_t > a_{t+1}$. For example, the arrangement $2, 3, 7, 1, 5, 4, 6, 8$ of the integers $1, 2, 3, 4, 5, 6, 7, 8$ has an internal peak at position 3, an internal peak at position 5, and no other internal peaks.
>
> **(a)** Determine the number of arrangements of $1, 2, 3, 4, 5$ that have an internal peak at position 3 and no other internal peaks.

> **(b)** Suppose that $n \geq 5$. Determine, in terms of n, an expression in closed form for the number of arrangements of $1, 2, \ldots, n-1, n$ that have an internal peak at position 2 and no other internal peaks.
>
> **(c)** Suppose that $n \geq 5$. Determine, in terms of n, an expression in closed form for the number of arrangements of $1, 2, \ldots, n-1, n$ that have an internal peak at position 3 and no other internal peaks.

(a) 注意：若位置 3 是内部峰，则其他位置不可能再有内部峰。取 $\binom{5}{3} = 10$ 种 $\{a_2, a_3, a_4\}$ 的组合；每种组合中最大数必在 a_3，其余两数在 $\{a_2, a_4\}$ 上有 2 种排法；剩下两数在 (a_1, a_5) 上有 2 种排法：

$$\binom{5}{3} \times 2 \times 2 = \mathbf{40}$$

(b) 要在位置 2 有内部峰，需 $a_1 < a_2 > a_3$ 且 $a_2 \geq 3$。由于 a_2 是唯一的峰，a_2 右侧 $(a_3, \ldots, a_n)$ 必为递减或 V 形。注意右侧最低点必为 $\{a_3, \ldots, a_n\}$ 中的最小值。

设 $a_2 = k \geq 3$，则 a_1 有 $k-1$ 种选择；谷底随之唯一确定，记为 a。对于 $(a,\ a_2)$ 区间内的数，每个可以放在左支或右支；大于 a_2 的数只能放在右支，且由于 a_2 是唯一峰，这些数的排列唯一确定。故给定 k 时共有 $(k-1) \cdot 2^{k-3}$ 种排列。

求和：

$$\sum_{k=3}^{n} (k-1) \cdot 2^{k-3} = \frac{1}{4} \sum_{k=3}^{n} (k-1) \cdot 2^{k-1} = \frac{1}{4} \sum_{l=2}^{n-1} l \cdot 2^{l} = \frac{1}{4}(n-2) \cdot 2^{n} = \mathbf{(n-2) \cdot 2^{n-}}$$

(c) 唯一峰在 a_3 时，$a_2 < a_3 > a_4$。设 $a_3 = k$，分两类（左侧为 V 形或递减）：

• $a_1 > a_2$：$(n-k)(k-1) \cdot 2^{k-3}$ 种；　• $a_1 < a_2$：$(k-1)(k-2) \cdot 2^{k-4}$ 种。

对 k 求和，两类合并：

$$\begin{aligned} &(n-3) \cdot 2 \cdot 2^0 + (n-4) \cdot 3 \cdot 2^1 + \cdots + 1 \cdot (n-2) \cdot 2^{n-4} \\ &\quad + 3 \cdot 2 \cdot 2^0 + 4 \cdot 3 \cdot 2^1 + \cdots + (n-1)(n-2) \cdot 2^{n-4} \\ &= n \cdot 2 \cdot 2^0 + n \cdot 3 \cdot 2^1 + \cdots + n \cdot (n-2) \cdot 2^{n-4} = \sum_{k=2}^{n-2} n \cdot k \cdot 2^{k-2} \\ &= \frac{n}{4} \sum_{k=2}^{n-2} k \cdot 2^{k} = \frac{n}{4} \cdot (n-3) \cdot 2^{n-1} = \mathbf{n(n-3) \cdot 2^{n-3}} \end{aligned}$$