Question	Answer	Marks
2(a)	$I_n = \left[-\frac{1}{3}e^{-3x}(1+3x)^n\right]_0^1 + n\int_0^1 e^{-3x}(1+3x)^{n-1}\,dx$	M1 A1
	$-\frac{1}{3}e^{-3}4^n + \frac{1}{3} + nI_{n-1} \Rightarrow 3I_n = 1 - 4^n e^{-3} + 3nI_{n-1}$	A1
		3

Question	Answer	Marks
2(b)	$I_0 = \int_0^1 e^{-3x}\,dx = -\frac{1}{3}\left[e^{-3x}\right]_0^1 = \frac{1}{3}\left(1-e^{-3}\right)$.	B1
	$I_1 = \frac{1}{3}\left(1-4e^{-3}+1-e^{-3}\right) = \frac{1}{3}\left(2-5e^{-3}\right)$ $I_2 = \frac{1}{3}\left(1-16e^{-3}+2\left(2-5e^{-3}\right)\right)$	M1
	$I_2 = \frac{1}{3}\left(5-26e^{-3}\right)$	A1
		3

Question	Answer	Marks	Guidance
1(a)	$f'(x) = -2xe^{-x^2}$	B1	Finds first derivative.
	$f''(x) = 4x^2e^{-x^2} - 2e^{-x^2}$	B1	Finds second derivative.
	$f(0) = 1 \quad f'(0) = 0 \quad f''(0) = -2$	M1	Evaluates derivatives at zero.
	$e^{-x^2} = 1 - x^2$	M1 A1	
		5	
1(b)	$\int_0^{\frac{1}{5}} 1 - x^2\,dx = \left[x - \frac{1}{3}x^3\right]_0^{\frac{1}{5}} = \frac{74}{375}$	M1 A1	Substitutes $1-x^2$ or better.
		2	

Question	Answer	Marks	Guidance
6(a)	$\frac{1}{n}\left(\frac{1}{\frac{1^2}{n^2}+1} + \frac{1}{\frac{2^2}{n^2}+1} + \ldots + \frac{1}{\frac{n^2}{n^2}+1}\right) = \frac{1}{n}\left(\frac{n^2}{1^2+n^2} + \frac{n^2}{2^2+n^2} + \ldots + \frac{n^2}{n^2+n^2}\right)$	M1 A1	Forms sum of the areas of the rectangles.
	$\int_0^1 \frac{1}{x^2+1}\,dx = \left[\tan^{-1}x\right]_0^1 = \frac{1}{4}\pi$	M1	Evaluates integral.
		A1	CAO
	$\frac{1}{n}\sum_{r=1}^{n}\frac{n^2}{n^2+r^2} = \sum_{r=1}^{n}\frac{n}{n^2+r^2} < \frac{1}{4}\pi$	A1	AG. Clear and convincing comparison with the integral.
		5	
6(b)	$\frac{1}{n}\left(\frac{1}{0^2+1} + \frac{1}{\frac{1^2}{n^2}+1} + \ldots + \frac{1}{\frac{(n-1)^2}{n^2}+1}\right) = \frac{1}{n}\left(\frac{n^2}{0^2+n^2} + \frac{n^2}{1^2+n^2} + \ldots + \frac{n^2}{(n-1)^2+n^2}\right)$	*M1 A1	Forms the sum of the areas of appropriate rectangles. M0 for more than n rectangles.
	$\frac{1}{n} + \sum_{r=1}^{n}\frac{n}{n^2+r^2} - \frac{1}{2n} \geq \frac{1}{4}\pi \Rightarrow \sum_{r=1}^{n}\frac{n}{n^2+r^2} \geq \frac{1}{4}\pi - \frac{1}{2n}$	DM1 A1	Compares with integral. Correctly adjust for 0^{th} and last rectangles. (Allow sign errors only for M1). Limits must match area covered by chosen rectangles.
		4	
6(c)	$\frac{1}{4}\pi$	B1	
		1	

Question	Answer	Marks	Guidance
3(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1+x^2\right)^n\right)=2nx^2\left(1+x^2\right)^{n-1}+\left(1+x^2\right)^n$	**M1 A1**	Uses the product rule to differentiate.
	$=2n\left(1+x^2-1\right)\left(1+x^2\right)^{n-1}+\left(1+x^2\right)^n$	***M1**	Uses $x^2=x^2+1-1$.
	$\left[x\left(1+x^2\right)^n\right]_0^1=2nI_n-2nI_{n-1}+I_n$	**DM1**	Integrates both sides using the limits given.
	$2^n=(2n+1)I_n-2nI_{n-1}\Rightarrow(2n+1)I_n=2^n+2nI_{n-1}$	**A1**	Substitutes limits and rearranges. AG.
		5	
3(b)	$I_{-1}=\left[\tan^{-1}x\right]_0^1=\frac{1}{4}\pi$	**B1**	
	$-I_{-1}=\frac{1}{2}-2I_{-2}$	**M1 A1**	Applies reduction formula with $n=-1$.
	$-\frac{1}{4}\pi=\frac{1}{2}-2I_{-2}\Rightarrow I_{-2}=\frac{1}{4}+\frac{1}{8}\pi$	**A1**	
		4	

Question	Answer	Marks	Guidance
3(a)	$A=2\pi\int_{\ln\frac{4}{3}}^{\ln\frac{12}{5}}\mathrm{e}^x\sqrt{1+\mathrm{e}^{2x}}\,\mathrm{d}x=2\pi\int_{\frac{4}{3}}^{\frac{12}{5}}u\sqrt{1+u^2}\left(\frac{1}{u}\right)\mathrm{d}u$	**M1A1**	Correct formula and applies substitution, AG.
		2	
3(b)	$u=\sinh v\Rightarrow\mathrm{d}u=\cosh v\,\mathrm{d}v$	**B1**	
	$A=2\pi\int_{\ln3}^{\ln5}\sqrt{1+\sinh^2 v}\cosh v\,\mathrm{d}v\quad A=2\pi\int_{\ln3}^{\ln5}\cosh^2 v\,\mathrm{d}v$	**M1A1**	Applies substitution and $\cosh^2 x=1+\sinh^2 x$. Need limits for A1.
	$\pi\int_{\ln3}^{\ln5}(\cosh 2v+1)\,\mathrm{d}v$	**M1**	Applies $2\cosh^2 x=\cosh 2x+1$.
	$=\pi\left[\frac{1}{2}\sinh 2v+v\right]_{\ln3}^{\ln5}$	**A1**	Correct integration.
	$\pi\left(\frac{312}{50}+\ln5-\frac{40}{18}-\ln3\right)=\pi\left(\frac{904}{225}+\ln\frac{5}{3}\right)$	**A1**	AG.
		6	

Question	Answer	Partial Marks	Guidance
6(a)	$\int_0^1(1-x)^2\,\mathrm{d}x<\left(\frac{1}{n}\right)\left(1-\frac{0}{n}\right)^2+\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^2+\cdots+\left(\frac{1}{n}\right)\left(1-\frac{n-1}{n}\right)^2$	**M1 A1**	Forms the sum of the areas of the rectangles. Must have first two terms and last term for A1. Accept $\left(\frac{1}{n}\right)\left(1-\frac{n}{n}\right)^2$ for the last term. Accept written in summation form with correct limits.
	$\frac{1}{n}+\frac{1}{n^3}\sum_{r=1}^{n-1}(n-r)^2=\frac{1}{n}+\frac{1}{n^3}\sum_{r=1}^{n-1}\left(n^2-2nr+r^2\right)$ $\left(=1-\frac{2}{n^2}\sum_{r=1}^{n-1}r+\frac{1}{n^3}\sum_{r=1}^{n-1}r^2\right)$	**M1 A1**	Setting up correct series so that formulae from MF19 can be applied. Can also recognise sum as $\frac{1}{n^3}\sum_{r=1}^{n}r^2$
	$\frac{1}{n}+\frac{n^2(n-1)}{n^3}-\frac{n^2(n-1)}{n^3}+\frac{n(n-1)(2n-1)}{6n^3}=\frac{1}{n}+\frac{2n^2-3n+1}{6n^2}=\frac{2n^2+3n+1}{6n^2}$	**A1**	AG. Must have gained previous accuracy mark.
		5	

Question	Answer	Partial Marks	Guidance
6(b)	$\int_0^1 (1-x)^2\,\mathrm{d}x > \left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(1-\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(1-\frac{n-1}{n}\right)^2$	**M1 A1**	Forms the sum of the areas of appropriate rectangles. Must have first two and last terms for A1. Accept $\left(\frac{1}{n}\right)\left(1-\frac{n}{n}\right)^2$ for the last term. Accept written in summation form with correct limits.
	$\frac{1}{n^3}\sum_{r=1}^{n-1}\left(n^2-2nr+r^2\right) = \frac{n^2(n-1)}{n^3} - \frac{n^2(n-1)}{n^3} + \frac{n(n-1)(2n-1)}{6n^3}$ $\left(=1-\frac{2}{n^2}\sum_{r=1}^{n} r + \frac{1}{n^3}\sum_{r=1}^{n} r^2\right)$	**M1**	Setting up correct series so that formulae from MF19 can be applied. Recognising sum as $\frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = \frac{1}{n^3}\sum_{r=1}^{n} r^2 - \frac{1}{n}$ without wrong working scores M1 A1 M1.
	$\frac{2n^2-3n+1}{6n^2}$	**A1**	
		4	
6(c)	$U_n - L_n = \frac{1}{n}$	**M1**	Simplifies $U_n - L_n$ to $\frac{c}{n}$.
	$\frac{1}{n} \to 0$ as $n \to \infty$	**A1**	AG.
		2	

Question	Answer	Marks	Guidance
3(a)	$\sum_{r=1}^{N} \frac{r}{2r^2-1} = 1 + \sum_{r=2}^{N} \frac{r}{2r^2-1}$	**M1 A1**	Compares with sum of the areas of the rectangles.
	$< 1 + \int_1^N \frac{x}{2x^2-1}\,\mathrm{d}x$	**M1**	Compares with integral.
	$\int_1^N \frac{x}{2x^2-1}\,\mathrm{d}x = \left[\frac{1}{4}\ln\left(2x^2-1\right)\right]_1^N$	**M1 A1**	Finds integral.
	$\sum_{r=1}^{N} \frac{r}{2r^2-1} < 1 + \frac{1}{4}\ln\left(2N^2-1\right)$	**M1 A1**	Inserts limits, AG.
		7	
3(b)	$\sum_{r=1}^{N} \frac{r}{2r^2-1} = \sum_{r=1}^{N-1} \frac{r}{2r^2-1} + \frac{N}{2N^2-1} > \int_1^N \frac{x}{2x^2-1}\,\mathrm{d}x + \frac{N}{2N^2-1}$	**M1 A1**	Compares with integral.
	$\frac{1}{4}\ln\left(2N^2-1\right) + \frac{N}{2N^2-1}$	**A1**	CWO.
	Alternative method for question 3(b)		
	$\sum_{r=1}^{N} \frac{r}{2r^2-1} > \int_1^{N+1} \frac{x}{2x^2-1}\,\mathrm{d}x$	**M1 A1**	Compares with integral.
	$\frac{1}{4}\ln\left(2(N+1)^2-1\right)$	**A1**	CWO.
		3	