Leave blank

3. (a) Show that the substitution $z = y^{-2}$ transforms the differential equation

$$\frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \quad \text{(I)}$$

into the differential equation

$$\frac{dz}{dx} - 4xz = -2xe^{-x^2} \quad \text{(II)}$$

(4)

(b) Solve differential equation (II) to find z as a function of x.

(5)

(c) Hence find the general solution of differential equation (I), giving your answer in the form $y^2 = f(x)$.

(1)

Leave blank

2. (a) Find the general solution of the differential equation

$$\left(x^2+1\right)\frac{dy}{dx} + xy - x = 0$$

giving your answer in the form $y = f(x)$.

(6)

(b) Find the particular solution for which $y = 2$ when $x = 3$

(2)

DO NOT WRITE IN THIS AREA

7.

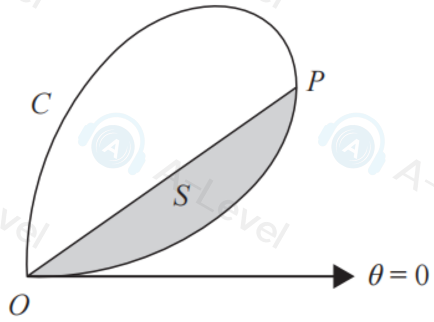

Figure 1

Figure 1 shows a sketch of curve C with polar equation

$$r = 3\sin 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

The point P on C has polar coordinates (R, ϕ). The tangent to C at P is perpendicular to the initial line.

(a) Show that $\tan\phi = \frac{1}{\sqrt{2}}$

(4)

(b) Determine the exact value of R.

(2)

The region S, shown shaded in Figure 1, is bounded by C and the line OP, where O is the pole.

(c) Use calculus to show that the exact area of S is

$$p\arctan\frac{1}{\sqrt{2}} + q\sqrt{2}$$

where p and q are constants to be determined.

Solutions relying entirely on calculator technology are not acceptable.

(7)

3: A complex number z is represented by the point P on an Argand diagram where $|z| = 1$

(a) Sketch the locus of P as z varies.

(1)

The transformation T from the z–plane, where $z = x + iy$, to the w–plane, where $w = u + iv$, is given by

$$w = \frac{9iz - i}{z + 1} \qquad z \neq -1$$

Given that the image under T of the locus of P in the z–plane, where $z \neq -1$, is the line l in the w–plane,

(b) determine, in simplest form, a Cartesian equation for l

(5)

Leave blank

5. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$ is given by

$$w = \frac{z+1}{z-3} \qquad z \neq 3$$

The straight line in the z-plane with equation $y = 4x$ is mapped by T onto the circle C in the w-plane.

(a) Show that C has equation

$$3u^2 + 3v^2 - 2u + v + k = 0$$

where k is a constant to be determined.

(5)

(b) Hence determine

(i) the coordinates of the centre of C

(ii) the radius of C

(2)

Leave blank

1. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{z + p\mathrm{i}}{\mathrm{i}z + 3} \qquad z \neq 3\mathrm{i} \quad p \in \mathbb{Z}$$

The point representing $\mathrm{i}(1 + \sqrt{3})$ is invariant under T.

Determine the value of p.

(3)

3. (a) Show that for $r \geqslant 1$

$$\frac{r}{\sqrt{r(r+1)}+\sqrt{r(r-1)}} \equiv A\left(\sqrt{r(r+1)}-\sqrt{r(r-1)}\right)$$

where A is a constant to be determined.

(2)

(b) Hence use the method of differences to determine a simplified expression for

$$\sum_{r=1}^{n} \frac{r}{\sqrt{r(r+1)}+\sqrt{r(r-1)}}$$

(3)

(c) Determine, as a surd in simplest form, the constant k such that

$$\sum_{r=1}^{n} \frac{kr}{\sqrt{r(r+1)}+\sqrt{r(r-1)}} = \sqrt{\sum_{r=1}^{n} r}$$

(2)

Leave blank

1. Use algebra to find the set of values of x for which

$$\frac{1}{x-2} > \frac{2}{x}$$

(5)

7. (a) Use De Moivre's theorem to

(i) show that

$$\sin 5\theta \equiv 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

(ii) determine an expression for $\cos 5\theta$ in terms of $\sin\theta$ and $\cos\theta$

(4)

(b) Hence show that, for $\cos 5\theta \neq 0$

$$\tan 5\theta \equiv \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

(2)

(c) Using the result of part (b) and showing all stages of your working, determine the solutions of the equation

$$2x^5 - 15x^4 - 20x^3 + 30x^2 + 10x - 3 = 0$$

giving your answers to 3 decimal places.

(5)

DO NOT WRITE IN THIS AREA

DO NOT

2.

$$x\frac{dy}{dx} - y^3 = 4$$

(a) Show that

$$x\frac{d^3y}{dx^3} = ay\left(\frac{dy}{dx}\right)^2 + \left(by^2 + c\right)\frac{d^2y}{dx^2}$$

where a, b and c are integers to be determined.

(4)

Given that $y = 1$ at $x = 2$

(b) determine the Taylor series expansion for y in ascending powers of $(x - 2)$, up to and including the term in $(x - 2)^3$, giving each coefficient in simplest form.

(3)

DO NOT WRITE IN THIS AREA

3. The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z}{z + 4\text{i}} \qquad z \neq -4\text{i}$$

The circle with equation $|z| = 3$ is mapped by T onto the circle C

Determine

(i) a Cartesian equation of C

(ii) the centre and radius of C

(8)