7.

$$y = \arccos(\operatorname{sech} x) \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \operatorname{sech} x$$

(3)

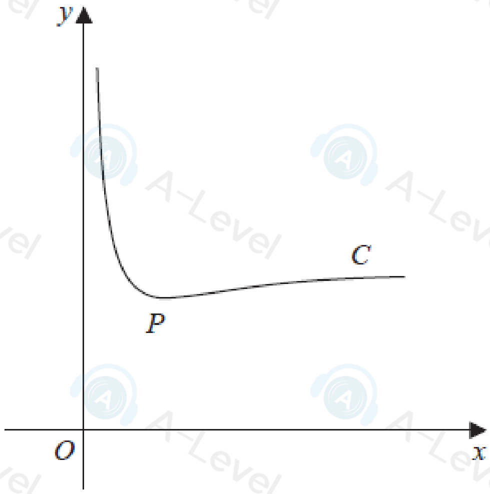

Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = f(x)$ where

$$f(x) = \arccos(\operatorname{sech} x) + \coth x \qquad x > 0$$

The point P is a minimum turning point of C

(b) Show that the x coordinate of P is $\ln\left(q + \sqrt{q}\right)$ where $q = \frac{1}{2}\left(1 + \sqrt{k}\right)$ and k is an integer to be determined.

(6)

6.

$$I_n = \int_0^{\sqrt{\frac{\pi}{2}}} x^n \cos(x^2)\,dx \qquad n \geqslant 1$$

(a) Prove that, for $n \geqslant 5$

$$I_n = \frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} - \frac{1}{4}(n-1)(n-3)I_{n-4}$$

(6)

(b) Hence, determine the exact value of I_5, giving your answer in its simplest form.

(3)

7. The curve C has parametric equations

$$x = \cosh t + t, \qquad y = \cosh t - t \qquad 0 \leqslant t \leqslant \ln 3$$

(a) Show that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\cosh^2 t$$

(3)

The curve C is rotated through 2π radians about the x-axis. The area of the curved surface generated is given by S.

(b) Show that

$$S = 2\pi\sqrt{2}\int_0^{\ln 3}\left(\cosh^2 t - t\cosh t\right)dt$$

(2)

(c) Hence find the value of S, giving your answer in the form

$$\frac{\pi\sqrt{2}}{9}(a + b\ln 3)$$

where a and b are constants to be determined.

(7)

6. The ellipse E has equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

The line l is the normal to E at the point $P(5\cos\theta, 3\sin\theta)$ where $0 < \theta < \frac{\pi}{2}$

(a) Using calculus, show that an equation for l is

$$5x\sin\theta - 3y\cos\theta = 16\sin\theta\cos\theta$$

(4)

Given that

- l intersects the y-axis at the point Q
- the midpoint of the line segment PQ is M

(b) determine the exact maximum area of triangle OMP as θ varies, where O is the origin.

You must justify your answer.

(5)

9. The ellipse E has equation

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

The point P lies on the ellipse and has coordinates $(5\cos\theta, 4\sin\theta)$ where $0 < \theta < \frac{\pi}{2}$

The line l is the normal to the ellipse at the point P.

(a) Show that an equation for l is

$$5x\sin\theta - 4y\cos\theta = 9\sin\theta\cos\theta$$

(5)

The point F is the focus of E that lies on the positive x-axis.

(b) Determine the coordinates of F.

(2)

The line l crosses the x-axis at the point Q.

(c) Show that

$$\frac{|QF|}{|PF|} = e$$

where e is the eccentricity of E.

(5)

7. A hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{25} = 1$$

where a is a positive constant.

The eccentricity of H is e.

(a) Determine an expression for e^2 in terms of a.

(1)

The line l is the directrix of H for which $x > 0$

The points A and A' are the points of intersection of l with the asymptotes of H.

(b) Determine, in terms of e, the length of the line segment AA'.

(3)

The point F is the focus of H for which $x < 0$

Given that the area of triangle AFA' is $\frac{164}{3}$

(c) show that a is a solution of the equation

$$30a^3 - 164a^2 + 375a - 4100 = 0$$

(4)

(d) Hence, using algebra and making your reasoning clear, show that the only possible value of a is $\frac{20}{3}$

(3)

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3.
$$y = \operatorname{arsinh}\left(\sqrt{x^2 - 1}\right) \qquad x > 1$$

(a) Prove that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - 1}}$

(3)

$$\mathrm{f}(x) = \frac{1}{3}\operatorname{arsinh}\left(\sqrt{x^2 - 1}\right) - \arctan x \qquad x > 1$$

(b) Determine the exact values of x for which $\mathrm{f}'(x) = 0$

(4)

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5. Given that

$$I_n = \int_0^{\frac{\pi}{4}} \cos^n \theta \, d\theta, \qquad n \geqslant 0$$

(a) prove that, for $n \geqslant 2$,

$$nI_n = \left(\frac{1}{\sqrt{2}}\right)^n + (n-1)I_{n-2}$$

(6)

(b) Hence find the exact value of I_5, showing each step of your working.

(5)

4. The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The line l is a normal to H at the point $P\,(a\sec\theta,\, b\tan\theta)$, $0 < \theta < \frac{\pi}{2}$

(a) Using calculus, show that an equation for l is

$$ax\sin\theta + by = (a^2 + b^2)\tan\theta$$

(5)

The line l meets the x-axis at the point Q, and the point M is the midpoint of PQ.

(b) Find the coordinates of M.

(3)

(c) Hence find the cartesian equation of the locus of M as θ varies, giving your answer in the form $y^2 = f(x)$.

(4)

3. (a) Given that $y = \operatorname{arsech}\left(\dfrac{x}{2}\right)$, where $0 < x \leqslant 2$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p}{x\sqrt{q - x^2}}$$

where p and q are constants to be determined.

(4)

In part (b) solutions based entirely on calculator technology are not acceptable.

$$\mathrm{f}(x) = \operatorname{artanh}(x) + \operatorname{arsech}\left(\frac{x}{2}\right) \qquad 0 < x \leqslant 1$$

(b) Determine, in simplest form, the exact value of x for which $\mathrm{f}'(x) = 0$

(5)

3.

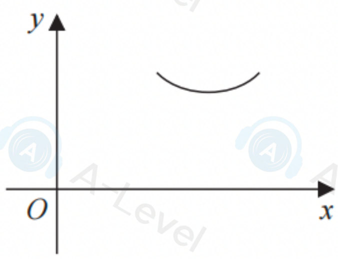

Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{1}{2}(\tan x + \cot x) \qquad \frac{\pi}{6} \leqslant x \leqslant \frac{\pi}{3}$$

(a) Show that the length of C is given by

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\tan^2 x + \cot^2 x\right)\mathrm{d}x$$

(6)

(b) Hence determine the exact length of C, giving your answer in simplest form.

(5)

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